ELLIPTIC INTEGRALS AND LANDEN'S TRANSFORMATION:

What I should've known for Geodesy

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Abstract

Fundamental problems in ellipsoidal geodesy (e.g. the direct and inverse problems of the geodesic; meridian distance; transverse Mercator projection) require, at some stage the evaluation of elliptic integrals. Usual methods – well documented in the geodetic literature – involve expanding the integrand into a series followed by term-by-term integration. This paper shows alternative approaches well known to mathematicians but little used by the geodetic community. They are based on a trigonometric transformation first discovered (in an algebraic form) by John Landen (1775) and used by A.M. Legendre (1786). An equivalent transformation was independently discovered and extensively used by Gauss (1818) in his algorithm of the arithmetic-geometric mean.

Introduction

Elliptic integrals arise naturally in expressions for the arc length of an ellipse and one of the earliest references to the length of an elliptic arc occurs in Kepler's *Astronomia nova* (Prague, 1609) announcing his discovery of the elliptical orbit of the Mars and wherein he offers some results to aid the calculation of the perimeter of an ellipse. This, and other early work on elliptic arcs is discussed by Dr G.N. Watson in his Presidential Address to the British Mathematical Association titled *The Marquis and the Land-Agent; A Tale of the Eighteenth Century* (Watson 1933) and we summarise his interesting address in the following paragraphs.

Watson describes early work by Newton (1642–1727) in 1676 and Maclaurin (1698–1746) in 1742 who both gave series for the quadrant length of the ellipse; each obtained from a binomial series of the differential arc length (a fluxion) followed by term-by-term integration (fluents). He also gives histories of the life and works of two other mathematicians Guilio Fagnano (1682-1766) and John Landen (1719-1790) – the *Marquis* and the *Land-Agent* – and their work on methods of calculating the arc lengths of the lemniscate, hyperbola and ellipse. Fagnano, born in Senigallia Italy and created a Marquis by the Pope in 1742 is famous for his discovery of a series of algebraic transformations of the parameters of the lemniscate that enable its length to be determined by a solution of quadratic equations. This is essentially the transformation of an integral into a form more amenable to solution by elementary methods.

Landen – born near Peterborough England and a land surveyor for twenty years before becoming the Earl Fitzwilliam's land-agent from 1762 to 1788 – used Fagnano's methods in a study of the hyperbola published in the *Philosophical Transactions* of the Royal Society 1775 and developed an algebraic transformation to obtain an expression for the arc of an hyperbola as the sum of two auxiliary elliptic arcs. Landen did not develop his transformation for other applications nor did he use it to evaluate elliptic integrals; but the French mathematician Adrien-Marie Legendre (1752–1833) defined a function that he called the elliptic integral of the First Kind (Legendre 1825, p.79); in our notation as

$$F\left(\phi,k\right)=\int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \tag{1}$$

And with $F\left(\psi,q\right)$ gave the connecting equations between the parameters ϕ,ψ and k,q

$$\sin\left(2\psi-\phi\right)=k\sin\phi\ ,\quad q=\frac{2\sqrt{k}}{1+k} \tag{2}$$

and stated the general relationship

$$F\left(\phi,k\right)=\frac{2}{1+k}F\left(\psi,q\right) \tag{3}$$

Equations (2) are now known as Landen's ascending transformation and repeated applications of (2) and (3) allow iterative schemes for evaluation of such integrals. We explain these schemes in following sections.

Legendre (1825, pp. 79-89) continued his development with another similar function that he called the elliptic integral of the Second Kind; in modern notation

$$E\left(\phi,k\right)=\int_0^\phi \sqrt{1-k^2\sin^2\theta}\,d\theta \tag{4}$$

And with $E\left(\psi,q\right)$ developed the relationship

$$E\left(\phi,k\right)+k\sin\phi=\left(1+k\right)E\left(\psi,q\right)+\left(1-k\right)F\left(\psi,q\right) \tag{5}$$

and concluded that this verified Landen's remarkable result for an arc of an hyperbola in terms of two elliptic arcs (Legendre 1825, p. 87).

Legendre did much work on elliptic functions and the classification of elliptic integrals and we adopt his notation in the following sections. Landen's transformation was independently discovered by Carl Friedrich Gauss (1777-1855) – called by him the *algorithm of the arithmetico-geometrical mean* – and used in connection with the evaluation of elliptic integrals arising from mass attractions in planetary theory.

Elliptic Integrals

Using the notation

$$\Delta=\Delta\left(\theta,k\right)=\sqrt{1-k^2\sin^2\theta} \tag{6}$$

the three elliptic integrals according to the classification of Legendre are

First Kind: $$F\left(\phi,k\right)=\int_0^\phi\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}=\int_0^\phi\frac{d\theta}{\Delta} \tag{7}$$

Second Kind: $$E\left(\phi,k\right)=\int_0^\phi\sqrt{1-k^2\sin^2\theta}\,d\theta=\int_0^\phi\Delta\,d\theta \tag{8}$$

Third Kind: $$\Pi\left(\phi,\alpha^2,k\right)=\int_0^\phi\frac{d\theta}{\left(1-\alpha^2\sin^2\theta\right)\sqrt{1-k^2\sin^2\theta}}=\int_0^\phi\frac{d\theta}{\left(1-\alpha^2\sin^2\theta\right)\Delta} \tag{9}$$

where ϕ is the *amplitude* $\left(0\le\phi\le\frac{1}{2}\pi\right)$; k is the *modulus* $\left(0\le k\le1\right)$ and α is a parameter $\left(-\infty<\alpha^2<\infty\right)$.

The cases where $\phi=\frac{1}{2}\pi$ are the *complete elliptic integrals*

$$\begin{aligned}K\left(k\right)&=F\left(\tfrac{1}{2}\pi,k\right)\\E\left(k\right)&=E\left(\tfrac{1}{2}\pi,k\right)\\\Pi\left(\alpha^2,k\right)&=\Pi\left(\tfrac{1}{2}\pi,\alpha^2,k\right)\end{aligned} \tag{10}$$

The case where $\alpha=k$ of the elliptic integral of the third kind is

$$\Pi\left(\phi,k\right)\equiv\Pi\left(\phi,k^2,k\right)=\int_0^\phi\frac{d\theta}{\left(\sqrt{1-k^2\sin^2\theta}\right)^3}=\int_0^\phi\frac{d\theta}{\Delta^3} \tag{11}$$

A useful result is

$$\frac{d}{d\theta}\left(\frac{\sin\theta\cos\theta}{\Delta}\right) = \frac{1 - 2\sin^2\theta + k^2\sin^4\theta}{\Delta^3} \tag{12}$$

from which we may write

$$\begin{aligned} k^2\frac{d}{d\theta}\left(\frac{\sin\theta\cos\theta}{\Delta}\right) &= \frac{1}{\Delta^3}\left(k^2 - 2k^2\sin^2\theta + k^4\sin^4\theta\right) \\ &= \frac{1}{\Delta^3}\left(1 - 2k^2\sin^2\theta + k^4\sin^4\theta + k^2 - 1\right) \\ &= \frac{1}{\Delta^3}\left(\Delta^4 + k^2 - 1\right) \end{aligned}$$

and hence

$$\frac{1-k^2}{\Delta^3} = \Delta - k^2\frac{d}{d\theta}\left(\frac{\sin\theta\cos\theta}{\Delta}\right) \tag{13}$$

Multiplying (11) by $\left(1-k^2\right)$ and using (13) gives

$$\begin{aligned} \left(1-k^2\right)\Pi\left(\phi,k\right) &= \int_0^\phi \frac{1-k^2}{\Delta^3}\,d\theta \\ &= \int_0^\phi \Delta\,d\theta - k^2\int_0^\phi \frac{d}{d\theta}\left(\frac{\sin\theta\cos\theta}{\Delta}\right)d\theta \\ &= \int_0^\phi \Delta\,d\theta - \frac{k^2\sin\phi\cos\phi}{\Delta\left(\phi,k\right)} \end{aligned}$$

and with (6) and (8) this becomes

$$\left(1-k^2\right)\Pi\left(\phi,k\right) = E\left(\phi,k\right) - \frac{k^2\sin\phi\cos\phi}{\sqrt{1-k^2\sin^2\phi}} \tag{14}$$

The Arithmetic-Geometric Mean (AGM)

Elliptic integrals cannot be evaluated in closed form by elementary methods; but they may be approximated (to a user defined degree of accuracy) by numerical methods that we will explain in the following sections. At the core of these methods is an iterative scheme that yields the arithmetic-geometric mean $M\left(a,b\right)$ of two real positive sequences $\left\{a_n\right\}$ and $\left\{g_n\right\}$

Assume $a \geq b > 0$ are real numbers. Put $a_0 = a$, $g_0 = b$; then define the sequences $\left\{a_n\right\}$ and $\left\{g_n\right\}$ (arithmetic and geometric means respectively) for all integers $n \geq 1$ as

$$\begin{aligned}
a_1 &= \tfrac{1}{2}\left(a_0 + g_0\right), & g_1 &= \sqrt{a_0 g_0} \\
a_2 &= \tfrac{1}{2}\left(a_1 + g_1\right), & g_2 &= \sqrt{a_1 g_1} \\
&\vdots \\
a_n &= \tfrac{1}{2}\left(a_{n-1} + g_{n-1}\right), & g_n &= \sqrt{a_{n-1} g_{n-1}}
\end{aligned} \tag{15}$$

The sequences $\left\{a_n\right\}$ and $\left\{g_n\right\}$ converge quadratically to a common limit $M\left(a,b\right)$ known as the *arithmetic-geometric mean* as shown below.

The *arithmetic-geometric inequality* is $A+B \geq 2\sqrt{AB}$; which is a direct consequence of $\left(\sqrt{A}-\sqrt{B}\right)^2 \geq 0$ for all real $A, B \geq 0$ (with equality when $A=B$). Using this, with $A = a_{n-1}$ and $B = g_{n-1}$ we obtain

$$a_{n-1} + g_{n-1} \geq 2\sqrt{a_{n-1} g_{n-1}} \tag{16}$$

That is, $\quad a_n \geq g_n$ for all $n \geq 1$

Further $\quad a_n = \tfrac{1}{2}\left(a_n + a_n\right) \geq \tfrac{1}{2}\left(a_n + g_n\right) = a_{n+1} \geq g_{n+1} = \sqrt{a_n g_n} \geq \sqrt{g_n g_n} = g_n$

So $\quad a = a_0 \geq a_1 \geq a_2 \geq \cdots \geq a_n \geq a_{n+1} \geq g_{n+1} \geq g_n \geq \cdots \geq g_2 \geq g_1 \geq g_0 = b$

Therefore the sequences $\left\{a_n\right\}$ and $\left\{g_n\right\}$ are each bounded, with $\left\{a_n\right\}$ non-increasing and $\left\{g_n\right\}$ non-decreasing. So each sequence is monotonically convergent.

Further $\quad a_n - g_n \leq a_n - g_{n-1} = \tfrac{1}{2}\left(a_{n-1} - g_{n-1}\right)$

And by induction $\quad a_n - g_n \leq \left(\tfrac{1}{2}\right)^n \left(a-b\right) \to 0$ as $n \to \infty$

So sequences $\left\{a_n\right\}$ and $\left\{g_n\right\}$ are convergent to the same limit denoted by $M\left(a,b\right)$ and $a_n \geq M\left(a,b\right) \geq g_n$ for all $n \geq 0$.

To analyse the convergence define $c_n^2 = a_n^2 - g_n^2$ then

$$\begin{aligned}
c_n^2 &= a_n a_n - g_n g_n \\
&= \tfrac{1}{2}\left(a_{n-1} + g_{n-1}\right)\tfrac{1}{2}\left(a_{n-1} + g_{n-1}\right) - \sqrt{a_{n-1} g_{n-1}}\sqrt{a_{n-1} g_{n-1}} \\
&= \tfrac{1}{4} a_{n-1}^2 + \tfrac{1}{4} g_{n-1}^2 - \tfrac{1}{2} a_{n-1} g_{n-1} \\
&= \tfrac{1}{4}\left(a_{n-1} - g_{n-1}\right)^2
\end{aligned}$$

So that $\quad 2c_n = a_{n-1} - g_{n-1} \qquad (17)$

which is a measure of the difference between the previous arithmetic and geometric means and hence a measure of the convergence of $\left\{a_n - g_n\right\}$.

Also $c_n^2 = \left(a_n + g_n\right)\left(a_n - g_n\right) = \left(2a_{n+1}\right)\left(2c_{n+1}\right)$ giving $c_{n+1} = \dfrac{c_n^2}{4a_{n+1}}$ and since $a_{n+1} \geq b$ we may write $c_{n+1} \leq \dfrac{c_n^2}{4b}$. This shows that $\left\{c_n\right\}$ converges to 0 quadratically.

Tables 1 and 2 show the iterations for $M\left(100,1\right)$ and $M\left(\sqrt{2},1\right)$ using (15) and (17). The terms a_n, g_n and c_n are shown to 30 decimal places.

n	a_n	g_n	c_n
0	100.000000000000000000000000000000	1.000000000000000000000000000000	
1	50.500000000000000000000000000000	10.000000000000000000000000000000	49.500000000000000000000000000000
2	30.250000000000000000000000000000	22.472205054244231864598140445491	20.250000000000000000000000000000
3	26.361102527122115932299070222745	26.072671571798851888825788640849	3.888897472877884067700929777255
4	26.216887049460483910562429431797	26.216490391739540869103263272023	0.144215477661632021736640790948
5	26.216688720600012389832846351910	26.216688719849834949125252818220	0.000198328860471520729583079887
6	26.216688720224923669479049585065	26.216688720224923669476366341013	0.000000000375088720353796766845
7	26.216688720224923669477707963039	26.216688720224923669477707963039	0.000000000000000000001341622026

M(a,b) = 26.216688720224923669477707963039   1/M(a,b) = 0.038143642420735908710117320873

Table 1. Arithmetic-Geometric mean $M\left(100,1\right)$

n	a_n	g_n	c_n
0	1.414213562373095048801688724210	1.000000000000000000000000000000	
1	1.207106781186547524400844362105	1.189207115002721066717499970560	0.207106781186547524400844362105
2	1.198156948094634295559172166333	1.198123521493120122606585571820	0.008949833091913228841672195772
3	1.198140234793877209082878869076	1.198140234677307205798383788190	0.000016713300757086476293297256
4	1.198140234735592207440631328633	1.198140234735592207439213655928	0.000000000058285001642247540443
5	1.198140234735592207439922492280	1.198140234735592207439922492280	0.000000000000000000000708836353

M(a,b) = 1.198140234735592207439922492280   1/M(a,b) = 0.834626841674073186281429732799

Table 2. Arithmetic-Geometric mean $M\left(\sqrt{2},1\right)$

$G = \dfrac{1}{M\left(1,\sqrt{2}\right)} = 0.834626841674073186281429732799$ is known as Gauss' constant[1] and the reciprocal of Gauss' constant is $M = \dfrac{1}{G} = 1.198140234735592207439922492280$ (Finch 2003, p. 420; Sloane A014549, A053004)

The following relationships between arithmetic-geometric means may be useful.

From (15) and deductions above, we can see

$$M\left(a,b\right) = M\left(a_n, g_n\right) \tag{18}$$

[1] In honour of C.F. Gauss (1777-1855) who computed $M\left(\sqrt{2},1\right)$ to 22 digits (by hand) in 1800 in the manuscript *De origine proprietatibusque generalibus numerorum mediorum arithmetic-geometricorum.* His work on the arithmetic-geometric mean was not published in his lifetime (Cox 1984)

Also for any constant λ

$$M\left(\lambda a,\lambda b\right)=\lambda M\left(a,b\right)\text{ so } aM\left(1,\frac{b}{a}\right)=bM\left(\frac{a}{b},1\right) \tag{19}$$

And by applying (18) we may write, with $a=1+x$, $b=1-x$

$$M\left(1+x,1-x\right)=M\left(1,\sqrt{1-x^2}\right) \tag{20}$$

And by applying (18) and (19) with $a=1$ and $b=x$

$$M\left(1,x\right)=M\left(\tfrac{1}{2}\left(1+x\right),\sqrt{x}\right)=\tfrac{1}{2}\left(1+x\right)M\left(1,\frac{2\sqrt{x}}{1+x}\right) \tag{21}$$

Legendre's form of the AGM

A variation of the arithmetic-geometric iteration known as Legendre's form is given as:

The modulus k and the complementary modulus k' are linked by

$$k^2+k'^2=1 \tag{22}$$

Defining
$$k^2=\frac{a^2-b^2}{a^2}=1-\frac{b^2}{a^2} \tag{23}$$

where $a>b$ are positive real numbers gives

$$k'=\sqrt{1-k^2}=\frac{b}{a} \tag{24}$$

and
$$\frac{1-k'}{1+k'}=\frac{a-b}{a+b} \tag{25}$$

Put $k'_0=b$ and define a sequence $\left\{k'_n\right\}$ as

$$k'_1=\frac{2\sqrt{k'_0}}{1+k'_0},\quad k'_2=\frac{2\sqrt{k'_1}}{1+k'_1},\quad \cdots,\quad k'_{n+1}=\frac{2\sqrt{k'_n}}{1+k'_n} \tag{26}$$

Then
$$M\left(1,b\right)=\prod_{n=0}^{\infty}\tfrac{1}{2}\left(1+k'_n\right) \tag{27}$$

The proof of this relationship is given by Jameson. Let $\left\{a_n\right\}$ and $\left\{g_n\right\}$ be the sequences generated by the iteration for $M\left(1,b\right)$ (with $a_0=1,\ g_0=b$) and let $k'_n=\frac{g_n}{a_n}$. Then $k'_0=b$ and

$$k'_{n+1} = \frac{g_{n+1}}{a_{n+1}} = \frac{2\sqrt{a_n g_n}}{a_n + g_n} = \frac{2a_n\sqrt{g_n/a_n}}{a_n\left(1+g_n/a_n\right)} = \frac{2\sqrt{k'_n}}{1+k'_n} \tag{28}$$

Also
$$\tfrac{1}{2}\left(1+k'_n\right) = \tfrac{1}{2}\left(1+\frac{g_n}{a_n}\right) = \frac{a_n+g_n}{2a_n} = \frac{a_{n+1}}{a_n} \tag{29}$$

so that $\tfrac{1}{2}\left(1+k'_0\right)\tfrac{1}{2}\left(1+k'_1\right)\tfrac{1}{2}\left(1+k'_2\right)\cdots\tfrac{1}{2}\left(1+k'_{n-1}\right) = \frac{a_1}{a_0}\frac{a_2}{a_1}\frac{a_3}{a_2}\cdots\frac{a_n}{a_{n-1}} = a_n$ (since $a_0 = 1$)

and hence
$$\prod_{r=0}^{n-1}\tfrac{1}{2}\left(1+k'_r\right) = a_n \to M\left(1,b\right) \quad \text{as } n\to\infty \tag{30}$$

The quadratic convergence of $\left\{k'_n\right\}$ to 1 is established by:

$$1-k'_{n+1} = \frac{1-2\sqrt{k'_n}+k'_n}{1+k'_n} < 1-2\sqrt{k'_n}+k'_n = \left(1-\sqrt{k'_n}\right)^2 < \left(1-k'_n\right)^2 \quad \text{since } 0<k'_n<1$$

Table 3 shows the iteration for the evaluation of $M\left(1,0.01\right)$ using (26) and (27)

n	a_n	g_n	k'_n
0	1.00000000000000000000000000000	0.01000000000000000000000000000	0.01000000000000000000000000000
1	0.50500000000000000000000000000	0.10000000000000000000000000000	0.19801980198019801980198020
2	0.30250000000000000000000000000	0.22472205054244231864598140445 5	0.74288281171055311949084761803 3
3	0.26361102527122115932299070222 7	0.26072671571798851888825788640 8	0.98905846388532094086377764339 0
4	0.26216887049460483910562429431 8	0.26216490391739540869103263272 0	0.99998487014418627537780811643 8
5	0.26216688720600012389832846351 9	0.26216688719849834949125252818 2	0.99999999997138549994995613316 6
6	0.26216688720224923669479049585 1	0.26216688720224923669476366341 0	0.99999999999999999999989765129 8
7	0.26216688720224923669477707963 0	0.26216688720224923669477707963 0	1.00000000000000000000000000000

M(a,b) = 0.262166887202249236694777079630 product[(1+kp)/2] = 0.262166887202249236694777079630

Table 3. Arithmetic-Geometric mean $M\left(1,0.01\right)$

A more efficient numerical evaluation of $M\left(1,b\right)$ in (27) can be made by first re-arranging the recurrence (26) as

$$\tfrac{1}{2}\left(1+k'_n\right) = \frac{\sqrt{k'_n}}{k'_{n+1}} = \frac{\sqrt{k'_n}}{\sqrt{k'_{n+1}}\sqrt{k'_{n+1}}} \tag{31}$$

Then, the right-hand-side of (27) can be approximated by

$$\tfrac{1}{2}\left(1+k_0\right)\tfrac{1}{2}\left(1+k'_1\right)\tfrac{1}{2}\left(1+k'_2\right)\cdots = \frac{\sqrt{k'_0}}{\sqrt{k'_1}\sqrt{k'_1}}\cdot\frac{\sqrt{k'_1}}{\sqrt{k'_2}\sqrt{k'_2}}\cdot\frac{\sqrt{k'_2}}{\sqrt{k'_3}\sqrt{k'_3}}\cdots = \sqrt{\frac{k'_0}{k'_1k'_2k'_3\cdots}} \tag{32}$$

giving
$$M\left(1,b\right) = \sqrt{\frac{k'_0}{k'_1k'_2k'_3\cdots}} \tag{33}$$

Another way to present this iteration follows from (22). If $t' = \dfrac{2\sqrt{k'}}{1+k'}$ and $t^2 + t'^2 = 1$

then $t^2 = 1 - t'^2 = 1 - \dfrac{4k'}{\left(1+k'\right)^2} = \dfrac{\left(1-k'\right)^2}{\left(1+k'\right)^2}$ and $t = \dfrac{1-k'}{1+k'}$.

So we have
$$k_{n+1} = \frac{1-k'_n}{1+k'_n} \quad \text{for all } n \geq 0 \tag{34}$$

hence $1 + k_{n+1} = 1 + \dfrac{1-k'_n}{1+k'_n} = \dfrac{2}{1+k'_n}$ so that (27) becomes

$$M\left(1,b\right) = \prod_{n=1}^{\infty} \frac{1}{1+k_n} \tag{35}$$

with
$$k'_n = \frac{2\sqrt{k'_{n-1}}}{1+k'_{n-1}} \quad \text{for all } n \geq 1 \text{ with starting value } k'_0 = b \tag{36}$$

Table 4 shows the iteration for the evaluation of $M\left(1,\dfrac{1}{\sqrt{2}}\right)$ using (35)

n	a_n	g_n	k_n
0	1.0000000000000000000000000000	0.707106781186547524400844362105	
1	0.853553390593273762200422181052	0.840896415253714543031125476233	0.171572875253809902396622551581
2	0.847224902923494152615773828643	0.847201266746891460403631453693	0.007469666729509581905511156011
3	0.847213084835192806509702641168	0.847213084752765366704298051780	0.000013949369424157397777847377
4	0.847213084793979086607000346474	0.847213084793979086605997900490	0.000000000048646226837637233590
5	0.847213084793979086606499123482	0.847213084793979086606499123482	0.000000000000000000000591613846

M(a,b) = 0.847213084793979086606499123482 product[1/(1+k)] = 0.847213084793979086606499123482

Table 4. Arithmetic-Geometric mean $M\left(1,\dfrac{1}{\sqrt{2}}\right)$

$M\left(1,\dfrac{1}{\sqrt{2}}\right) = \dfrac{M}{\sqrt{2}} = \dfrac{1}{G\sqrt{2}} = 0.847213084793979086606499123482$ is known as the Ubiquitous Constant (Finch 2003, p. 421; Sloane A096427).

Landen's Transformation

Landen's transformation may be developed by considering the geometric relationships between two angles θ and ω and a parameter k $\left(0 \leq k \leq 1\right)$. These relationships are shown in Figure 1.

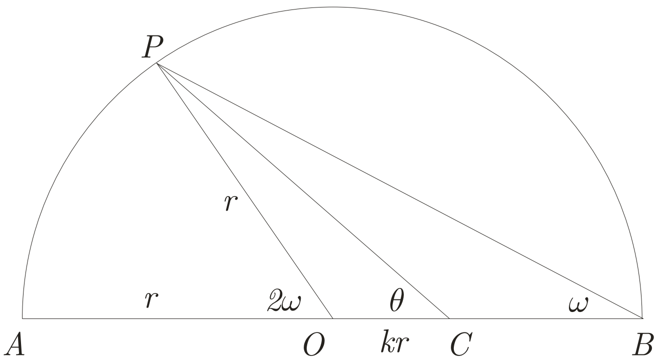

Figure 1

In Figure 1:

(i) AB is a diameter of a circle of radius r and centre O, $AO = OB = OP = r$;

(ii) C is a point on OB such that $OC = kr$ with $0 \le k \le 1$;

(iii) $P\hat{B}C = \omega$; $P\hat{C}O = \theta$; $P\hat{O}A = 2\omega$ (property of chord AP and circumferential angle ω and central angle 2ω); $O\hat{P}C = 2\omega - \theta$ and $\omega \le \theta$ because $\theta = \omega + C\hat{P}B \ge \omega$

Using the sine rule in triangle OPC we obtain

$$\sin\left(2\omega - \theta\right) = k \sin\theta \tag{37}$$

We now proceed with a sequence of manipulations to firstly obtain a useful differential relationship linking $d\theta$ and $d\omega$ [see (38)] and then simplifying this equation into one where the functions on the left-hand and the right-hand-sides are similar in form and their variables linked by simple relationships [see (49) and (53)].

Let $y = 2\omega - \theta$ in (37) then, using the chain rule, differentiate with respect to θ; giving $\cos y \, dy = k \cos\theta \, d\theta$. With $dy = 2\,d\omega - d\theta$ we write

$$\cos\left(2\omega - \theta\right)\left(2\,d\omega - d\theta\right) = k\cos\theta\,d\theta$$

And this can be re-arranged as

$$\frac{d\theta}{\cos\left(2\omega - \theta\right)} = \frac{2\,d\omega}{\cos\left(2\omega - \theta\right) + k\cos\theta} \tag{38}$$

Squaring both sides of (37), then subtracting the result from unity and using $\cos^2 A = 1 - \sin^2 A$ we may write

$$\cos\left(2\omega - \theta\right) = \sqrt{1 - k^2 \sin^2\theta} \tag{39}$$

Expanding (37) with the aid of $\sin\left(A-B\right)=\sin A\cos B-\cos A\sin B$ gives

$$\sin 2\omega\cos\theta-\cos 2\omega\sin\theta=k\sin\theta \tag{40}$$

and dividing both sides of (40) by $\cos\theta$ and re-arranging gives

$$\tan\theta=\frac{\sin 2\omega}{k+\cos 2\omega} \tag{41}$$

This relationship is represented by the right-angled triangle in Figure 2, noting that its hypotenuse is obtained as

$$\sqrt{\sin^2 2\omega+\left(k+\cos 2\omega\right)^2}=\sqrt{\sin^2 2\omega+\cos^2 2\omega+2k\cos 2\omega+k^2}=\sqrt{1+2k\cos 2\omega+k^2}$$

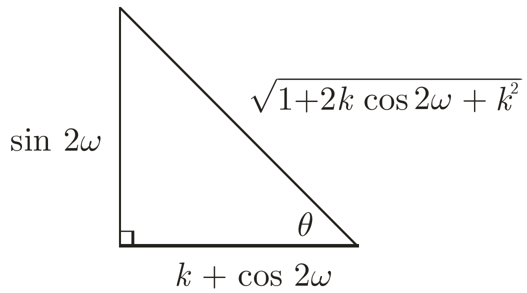

Figure 2

Using Figure 2 we have

$$\sin\theta=\frac{\sin 2\omega}{\sqrt{1+2k\cos 2\omega+k^2}} \tag{42}$$

$$\cos\theta=\frac{k+\cos 2\omega}{\sqrt{1+2k\cos 2\omega+k^2}} \tag{43}$$

and multiplying (42) and (43) by $\sin\theta$ and $\cos\theta$ respectively and re-arranging gives

$$\sin^2\theta\sqrt{1+2k\cos 2\omega+k^2}=\sin 2\omega\sin\theta \tag{44}$$

$$\cos^2\theta\sqrt{1+2k\cos 2\omega+k^2}=\left(k+\cos 2\omega\right)\cos\theta \tag{45}$$

With the aid of $\cos\left(A-B\right)=\cos A\cos B+\sin A\sin B$ and (44) and (45) the denominator of the right-hand-side of (38) may be written as

$$\begin{aligned}\cos\left(2\omega-\theta\right)+k\cos\theta&=\left(k+\cos 2\omega\right)\cos\theta+\sin 2\omega\sin\theta\\&=\sqrt{1+2k\cos 2\omega+k^2}\end{aligned} \tag{46}$$

Note that using $\cos 2A=1-2\sin^2 A$ we may write

$$\begin{aligned}1+2k\cos 2\omega+k^2&=1+2k\left(1-2\sin^2\omega\right)+k^2\\&=1+2k+k^2-4k\sin^2\omega\\&=\left(1+k\right)^2-4k\sin^2\omega\\&=\left(1+k\right)^2\left\{1-\frac{4k}{\left(1+k\right)^2}\sin^2\omega\right\}\end{aligned}\tag{47}$$

So using (46) and (47) we have

$$\cos\left(2\omega-\theta\right)+k\cos\theta=\left(1+k\right)\sqrt{1-\frac{4k}{\left(1+k\right)^2}\sin^2\omega}\tag{48}$$

and substituting (39) and (48) into (38) gives (Rösch 2011, eq. (9))

$$\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}=\frac{2}{1+k}\frac{d\omega}{\sqrt{1-\dfrac{4k}{\left(1+k\right)^2}\sin^2\omega}}\tag{49}$$

Now, since $0\le k\le 1$ (see Figure 1 and the explanation) we use the rules for inequalities in the following development:

$$1+k\le 2,\quad \left(1+k\right)^2\le 4,\quad 1\le\frac{4}{\left(1+k\right)^2},\quad k\le\frac{4k}{\left(1+k\right)^2}\tag{50}$$

Defining

$$q=\frac{2\sqrt{k}}{1+k}\tag{51}$$

and so $k^2\le k\le\dfrac{4k}{\left(1+k\right)^2}=q^2$ then, taking positive square roots,

$$k\le q\tag{52}$$

Using (51) and (7) we may write (49) as

$$F\left(\phi,k\right)=\frac{2}{1+k}F\left(\psi,q\right)\tag{53}$$

where q and k are related by (51) with $q\ge k$ and from (37) (noting Figure 1 with $\psi\equiv\omega$ and $\phi\equiv\theta$), we have

$$\sin\left(2\psi-\phi\right)=k\sin\phi\tag{54}$$

with $\psi<\phi$

So the left-hand-side of (53) has been replaced by a similar function (with a multiplier $\frac{2}{1+k}$) with smaller amplitude ψ and a larger modulus q.

We summarise Legendre's result as

$$\boxed{\begin{array}{ll} \text{If} & F\left(\phi,k\right)=\int_0^{\phi}\dfrac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \\ \text{and} & \sin\left(2\psi-\phi\right)=k\sin\phi\,,\ q=\dfrac{2\sqrt{k}}{1+k}\ \text{ where } \psi<\phi\,,\ q\geq k \\ \text{then} & F\left(\phi,k\right)=\dfrac{2}{1+k}F\left(\psi,q\right) \end{array}}$$

These results are now used to develop an iterative scheme to evaluate an elliptic integral of the first kind.

Evaluating Elliptic Integrals of the First Kind

Defining $k=k_0$, $q=k_1$ and $\phi=\phi_0$, $\psi=\phi_1$ we may write (53) as

$$F\left(\phi_0,k_0\right)=\frac{2}{1+k_0}F\left(\phi_1,k_1\right) \tag{55}$$

and the right-hand-side of (55) can be iterated as

$$\begin{aligned} &\frac{2}{1+k_0}\cdot\frac{2}{1+k_1}F\left(\phi_2,k_2\right) \\ =&\frac{2}{1+k_0}\cdot\frac{2}{1+k_1}\cdot\frac{2}{1+k_2}F\left(\phi_3,k_3\right) \end{aligned}$$

giving the sequence (Rösch 2011, eq. (14))

$$\boxed{F\left(\phi_0,k_0\right)=\frac{2}{1+k_0}\cdot\frac{2}{1+k_1}\cdot\frac{2}{1+k_2}\cdots\frac{2}{1+k_{n-1}}F\left(\phi_n,k_n\right)} \tag{56}$$

where the moduli $\left\{k_n\right\}$ and amplitudes $\left\{\phi_n\right\}$ are obtained from the recurrence relationships

$$k_{n+1}=\frac{2\sqrt{k_n}}{1+k_n}\quad\text{with starting value } k_0 \text{ and } k_{n+1}\geq k_n \tag{57}$$

$$\sin\left(2\phi_{n+1}-\phi_n\right)=k_n\sin\phi_n\quad\text{with starting values } k_0,\phi_0 \text{ and } \phi_{n+1}<\phi_n \tag{58}$$

Equations (57) and (58) are *Landen's Ascending Transformation* (DLMF, §19.8)

Note that as $n\to\infty$ the sequence of moduli $\left\{k_n\right\}$ converges to unity (see the Arithmetic Geometric Mean section) and $0\leq k_n\leq 1$. Also $0\leq\phi_n\leq\frac{1}{2}\pi$ and consequently

$\sin\left(2\phi_{n+1} - \phi_n\right) = k_n \sin\phi_n \le \sin\phi_n$ and $2\phi_{n+1} \le 2\phi_n$ so the sequence of amplitudes $\left\{\phi_n\right\}$ is monotonically decreasing, bounded below by zero, and hence convergent.

Let $\lim_{n\to\infty}\phi_n = \hat{\phi}$ then with the aid of the standard result $\int \sec\theta\, d\theta = \ln\tan\left(\frac{1}{4}\pi + \frac{1}{2}\theta\right)$ we may write

$$\lim_{n\to\infty} F\left(\phi_n, k_n\right) = F\left(\hat{\phi}, 1\right) = \int_0^{\hat{\phi}} \frac{d\theta}{\sqrt{1-\sin^2\theta}} = \int_0^{\hat{\phi}} \sec\theta\, d\theta = \ln\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\hat{\phi}\right) \tag{59}$$

Using this result in (56) we may write a formula for the evaluation of the elliptic integral of the first kind as

$$F\left(\phi_0, k_0\right) = \frac{2}{1+k_0}\cdot\frac{2}{1+k_1}\cdot\frac{2}{1+k_2}\cdots \ln\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\hat{\phi}\right) \tag{60}$$

A more efficient numerical evaluation of $F\left(\phi_0, k_0\right)$ in (60) can made by first re-arranging the recurrence (57) as

$$\frac{2}{1+k_n} = \frac{k_{n+1}}{\sqrt{k_n}} = \frac{\sqrt{k_{n+1}}\sqrt{k_{n+1}}}{\sqrt{k_n}}$$

and in the right-hand-side of (60) we may write

$$\frac{2}{1+k_0}\cdot\frac{2}{1+k_1}\cdot\frac{2}{1+k_2}\cdots\frac{2}{1+k_{n-1}} = \frac{\sqrt{k_1}\sqrt{k_1}}{\sqrt{k_0}}\cdot\frac{\sqrt{k_2}\sqrt{k_2}}{\sqrt{k_1}}\cdot\frac{\sqrt{k_3}\sqrt{k_3}}{\sqrt{k_2}}\cdots\frac{\sqrt{k_n}\sqrt{k_n}}{\sqrt{k_{n-1}}} = \sqrt{k_n}\sqrt{\frac{k_1 k_2 k_3 \cdots k_n}{k_0}}$$

And with the approximations $\hat{\phi} \approx \phi_n$ and $k_n \approx 1$ (60) can be approximated as (King 1924, eq. (8))

$$\boxed{F\left(\phi_0, k_0\right) \approx \sqrt{\frac{k_1 k_2 k_3 \cdots k_n}{k_0}}\ \ln\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi_n\right)} \tag{61}$$

Using (61) with the recurrence relationships (57) and (58) elliptic integrals of the first kind can be easily evaluated. For example, Table 5 shows the sequences $\left\{k_n\right\}$ and $\left\{\phi_n\right\}$ for $n = 0,1,2,\ldots,7$ for the evaluation of $F\left(\phi_0, k_0\right) = F\left(\frac{1}{3}\pi, 0.08\right)$ using Landen's ascending transformation.

n	k_n	ϕ_n (degrees)
0	0.080000000000000000000000000000	60.000000000000000000000000000000
1	0.523782800878924092148773601559	31.986375290275345206426642287209
2	0.949910187049589210997543775191	24.047430982645159265462549335921
3	0.999670002027321953622743219568	23.410227894528127587073492551888
4	0.999999986383174006610440236550	23.406135016316805818730472567793
5	0.999999999999999997682275591761 8	23.406134847458754170823337386119
6	1.000000000000000000000000000000	23.406134847458753883409497070244
7	1.000000000000000000000000000000	23.406134847458753883409497070244

Table 5. Landen's ascending transformation

Sequences $\left\{k_n\right\}$ and $\left\{\phi_n\right\}$ for $F\left(\frac{1}{3}\pi, 0.08\right)$

For $n = 7$ the numerical values (30 decimal places) for (61) are

$$
\begin{aligned}
\sqrt{\frac{k_1 k_2 k_3 \cdots k_7}{k_0}} &= 2.493447468386180336582176623321 \\
\phi_7 &= 23.406134847458753883409497070244 \text{ degrees} \\
\ln \tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi_7\right) &= 0.420374825518291193367606198921 \\
F\left(\tfrac{1}{3}\pi, 0.08\right) &\approx 1.048182544461865455398483533571
\end{aligned}
$$

A Second Method of Evaluating Elliptic Integrals of the First Kind

Consider a re-arrangement of (53) as

$$
\boxed{F\left(\psi, q\right) = \tfrac{1}{2}\left(1 + k\right) F\left(\phi, k\right) \qquad (62)}
$$

where an elliptic integral with a given amplitude ψ and modulus q on the left-hand-side is replaced by an elliptic integral on the right-hand-side with a larger amplitude ϕ and smaller modulus k.

The relationships between ψ, q (known) and ϕ, k (unknown) can be obtained as follows:

(A) expression for k given q

From (51) write

$$
1 - q^2 = 1 - \frac{4k}{\left(1+k\right)^2} = \frac{\left(1+k\right)^2 - 4k}{\left(1+k\right)^2} = \frac{1 - 2k + k^2}{\left(1+k\right)^2} = \frac{\left(1-k\right)^2}{\left(1+k\right)^2} \tag{63}
$$

The modulus q and complementary modulus q' are linked by

$$
q^2 + q'^2 = 1 \tag{64}
$$

and using this relationship and (63) we write

$$q' = \sqrt{1-q^2} = \frac{1-k}{1+k} \tag{65}$$

Re-arranging (65) and solving for k gives gives

$$k = \frac{1-q'}{1+q'} = \frac{1-\sqrt{1-q^2}}{1+\sqrt{1-q^2}} \tag{66}$$

(B) expression for ϕ given ψ

Using (37) with $\phi = \theta, \psi = \omega$ and $\sin\left(2\omega - \theta\right) = -\sin\left(\theta - 2\omega\right) = -\sin\left(\left(\phi - \psi\right) - \psi\right)$ we write

$$-\sin\left(\left[\phi - \psi\right] - \psi\right) = k\sin\left(\left[\phi - \psi\right] + \psi\right)$$

and using $\sin\left(A \pm B\right) = \sin A \cos B \pm \sin B \cos A$ and re-arranging gives

$$\begin{aligned} \sin\psi\cos\left(\phi - \psi\right) - k\sin\psi\cos\left(\phi - \psi\right) &= k\cos\psi\sin\left(\phi - \psi\right) + \cos\psi\sin\left(\phi - \psi\right) \\ \left(1-k\right)\sin\psi\cos\left(\phi - \psi\right) &= \left(1+k\right)\cos\psi\sin\left(\phi - \psi\right) \end{aligned}$$

and

$$\tan\left(\phi - \psi\right) = \frac{1-k}{1+k}\tan\psi \tag{67}$$

Substituting (65) into (67) gives

$$\tan\left(\phi - \psi\right) = \sqrt{1-q^2}\,\tan\psi \tag{68}$$

Defining $\psi = \phi_0$, $\phi = \phi_1$, and $q = k_0$, $k = k_1$ we write (62) as

$$F\left(\phi_0, k_0\right) = \tfrac{1}{2}\left(1 + k_1\right)F\left(\phi_1, k_1\right) \tag{69}$$

where $k_1 < k_0$ and $\phi_1 > \phi_0$

The right-hand-side of (69) can be iterated as

$$\begin{aligned} &\tfrac{1}{2}\left(1 + k_1\right)\tfrac{1}{2}\left(1 + k_2\right)F\left(\phi_2, k_2\right) \\ &= \tfrac{1}{2}\left(1 + k_1\right)\tfrac{1}{2}\left(1 + k_2\right)\tfrac{1}{2}\left(1 + k_3\right)F\left(\phi_3, k_3\right) \end{aligned}$$

giving the sequence (Rösch 2011, eq. (22))

$$\boxed{F\left(\phi_0, k_0\right) = \tfrac{1}{2}\left(1 + k_1\right)\tfrac{1}{2}\left(1 + k_2\right)\tfrac{1}{2}\left(1 + k_3\right)\cdots\tfrac{1}{2}\left(1 + k_n\right)F\left(\phi_n, k_n\right)} \tag{70}$$

where the moduli $\left\{k_n\right\}$ and amplitudes $\left\{\phi_n\right\}$ are obtained from the recurrence relationships

$$k_{n+1} = \frac{1-\sqrt{1-k_n^2}}{1+\sqrt{1-k_n^2}} \quad \text{with starting value } k_0 \tag{71}$$

$$\tan\left(\phi_{n+1}-\phi_n\right) = \sqrt{1-k_n^2}\tan\phi_n \quad \text{with starting values } k_0 \text{ and } \phi_0 \tag{72}$$

Equations (71) and (72) are *Landen's Descending Transformation* (DLMF, §19.8)

Now $k_n = \dfrac{1-\sqrt{1-k_{n-1}^2}}{1+\sqrt{1-k_{n-1}^2}} \cdot \dfrac{1+\sqrt{1-k_{n-1}^2}}{1+\sqrt{1-k_{n-1}^2}} = \dfrac{k_{n-1}^2}{\left(1+\sqrt{1-k_{n-1}^2}\right)^2}$ and by re-arrangement we may write $k_{n-1} = \sqrt{k_n}\left(1+\sqrt{1-k_{n-1}^2}\right)$ and using (71) we have

$$\begin{aligned} k_n - k_{n-1} &= \frac{1-\sqrt{1-k_{n-1}^2}}{1+\sqrt{1-k_{n-1}^2}} - \sqrt{k_n}\left(1+\sqrt{1-k_{n-1}^2}\right) \\ &= \frac{1-\sqrt{1-k_{n-1}^2} - \sqrt{k_n}\left(1+\sqrt{1-k_{n-1}^2}\right)^2}{1+\sqrt{1-k_{n-1}^2}} \\ &= \frac{1-\sqrt{1-k_{n-1}^2} - k_{n-1}\left(1+\sqrt{1-k_{n-1}^2}\right)}{1+\sqrt{1-k_{n-1}^2}} \end{aligned} \tag{73}$$

The numerator of (73) can be written as

$$1-k_{n-1}-\left(1+k_{n-1}\right)\sqrt{1-k_{n-1}^2} = \sqrt{1-k_{n-1}}\left\{\sqrt{1-k_{n-1}}-\left(1+k_{n-1}\right)\sqrt{1+k_{n-1}}\right\} < 0$$

since the term in braces will be less than zero; and using this result in (73) allows us to write $k_n < k_{n-1}$ for all n. And $0 < k_n < 1$ for all n.

Thus the sequence $\left\{k_n\right\}$ is monotonically decreasing, bounded below by zero, and hence convergent, and $\lim_{n\to\infty} k_n = 0$.

Suppose $\lim_{n\to\infty}\phi_n = \hat{\phi}$, then $\lim_{n\to\infty} F\left(\phi_n, k_n\right) = F\left(\hat{\phi}, 0\right) = \int_0^{\hat{\phi}} \dfrac{d\theta}{\sqrt{1-0}} = \int_0^{\hat{\phi}} d\theta = \hat{\phi}$, therefore Landen's descending transformation (70) becomes

$$\boxed{F\left(\phi_0, k_0\right) = \hat{\phi}\prod_{n=1}^{\infty}\tfrac{1}{2}\left(1+k_n\right)} \tag{74}$$

The sequences $\left\{k_n\right\}$ and $\left\{\phi_n\right\}$ are obtained from the recurrence relationships (71) and (72).

For the purposes of evaluation we might truncate (74) to

$$\boxed{F\left(\phi_0, k_0\right) \approx \phi_n \prod_{r=1}^{n} \tfrac{1}{2}\left(1 + k_r\right)} \qquad (75)$$

Table 6 shows the sequences $\left\{k_n\right\}$ and $\left\{\phi_n\right\}$ for $n = 0, 1, 2, \ldots, 5$ for the evaluation of $F\left(\tfrac{1}{3}\pi, 0.08\right)$ using Landen's descending transformation

n	k_n	ϕ_n *(degrees)*
0	0.080000000000000000000000000000	60.000000000000000000000000000000
1	0.001605140572193028090147065969	119.920289644141316680103929766278
2	0.000000644119893905114137330012	239.840611197798147947606152992362
3	0.000000000000103722609431105390	479.681222395591132768280569471756
4	0.000000000000000000000000002690	959.362444791182265536561139076104
5	0.000000000000000000000000000000	1918.724889582364531073122278152207

Table 6. Landen's descending transformation

Sequences $\left\{k_n\right\}$ and $\left\{\phi_n\right\}$ for $F\left(\tfrac{1}{3}\pi, 0.08\right)$

For $n = 5$ the numerical values (30 decimal places) for (75) are

$$\begin{aligned} \prod_{r=1}^{5} \tfrac{1}{2}\left(1 + k_r\right) &= 0.031300180803940431169805270126 \\ \phi_5 &= 1918.724889582364531073122278152207 \text{ degrees} \\ F\left(\tfrac{1}{3}\pi, 0.08\right) &\approx 1.048182544461865455398483533571 \end{aligned}$$

Note that ϕ_{n+1} is almost twice ϕ_n as the iterative scheme progresses. This can be seen from the recurrence relationship (72) where $\phi_{n+1} = \arctan\left(\sqrt{1 - k_n^2}\,\tan\phi_n\right) + \phi_n$.

Noting that for large *n*, $k_n \approx 0$, so $\sqrt{1 - k_n^2} \approx 1$. Hence
$\phi_{n+1} = \arctan\left(\sqrt{1 - k_n^2}\,\tan\phi_n\right) + \phi_n \approx \arctan\left(\tan\phi_n\right) + \phi_n = 2\phi_n$. In fact $\phi_{n+1} < 2\phi_n$.

Also, computer implementations of $\theta = \arctan x$ where $-\infty < x < \infty$ return the principal value $-\tfrac{1}{2}\pi < \theta < \tfrac{1}{2}\pi$.

For example, in Table 6 $\phi_2 = \phi_{n+1}$ is evaluated as $x = \sqrt{1 - k_1^2}\,\tan\phi_1 = -1.7376268469\ldots$ but $\arctan x$ returns $\theta = -60.079678446\ldots$ deg whereas we require $\phi = \pi + \theta = 119.920321553\ldots$ to be returned to give $\phi_2 = 119.920321553\ldots + 119.92028964\ldots = 239.84061119779\ldots$ deg. So multiples of π must be accumulated as the iteration proceeds. This can be done with the aid of a *ceiling* function common to most computer languages where *ceiling(x)* rounds x upwards to the next integer and the appropriate multiples of π are obtained from:

$$ceiling\left(\frac{\phi_n}{\pi} - 0.5\right)\times\pi$$

A Third Method of Evaluating Elliptic Integrals of the First Kind

Remembering the definition of k^2 and the relationships between k, k' (modulus and complimentary modulus) and the positive real numbers $a \geq b$ given by (22) to (25) we substitute (23) into (6) to obtain

$$\begin{aligned}\Delta &= \sqrt{1-\left(1-\frac{b^2}{a^2}\right)\sin^2\theta} \\ &= \sqrt{1-\sin^2\theta+\frac{b^2}{a^2}\sin^2\theta} \\ &= \frac{1}{a}\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}\end{aligned} \tag{76}$$

Substituting (76) into (7) gives

$$F\left(\phi,k\right) = \int_0^\phi \frac{d\theta}{\Delta} = a\int_0^\phi \frac{d\theta}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}} \tag{77}$$

Defining

$$I\left(\phi,a,b\right) = \int_0^\phi \frac{d\theta}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}} \tag{78}$$

we write (77) as

$$\boxed{F\left(\phi,k\right) = a\, I\left(\phi,a,b\right)} \tag{79}$$

Now, using (62) we write

$$F\left(\phi,k\right) = \tfrac{1}{2}\left(1+u\right)F\left(\beta,u\right) \tag{80}$$

where $\beta > \phi,\ u < k$.

Now with the aid of (62) to (68) and (22) to (25) we write

$$u = \frac{1-k'}{1+k'} = \frac{a-b}{a+b} \tag{81}$$

$$\tan\left(\beta-\phi\right) = k'\tan\phi \tag{82}$$

Using (24), (81) and some algebra we have $\tfrac{1}{2}\left(1+u\right)=\dfrac{1}{1+k'}=\dfrac{a}{a+b}$ and substituting these results into (80) gives

$$F\left(\phi,k\right)=\frac{a}{a+b}F\left(\beta,\frac{a-b}{a+b}\right) \tag{83}$$

But using (7)

$$\begin{aligned}
F\left(\beta,\frac{a-b}{a+b}\right) &= \int_0^\beta \frac{d\theta}{\sqrt{1-\dfrac{\left(a-b\right)^2}{\left(a+b\right)^2}\sin^2\theta}} \\
&= \left(a+b\right)\int_0^\beta \frac{d\theta}{\sqrt{\left(a+b\right)^2-\left(a-b\right)^2\sin^2\theta}} \\
&= \left(a+b\right)\int_0^\beta \frac{d\theta}{2\sqrt{\left(\tfrac{1}{2}\left(a+b\right)\right)^2\left(\cos^2\theta+\sin^2\theta\right)-\left(\tfrac{1}{2}\left(a-b\right)\right)^2\sin^2\theta}} \\
&= \tfrac{1}{2}\left(a+b\right)\int_0^\beta \frac{d\theta}{\sqrt{\left(\tfrac{1}{2}\left(a+b\right)\right)^2\cos^2\theta+\left(\sqrt{ab}\right)^2\sin^2\theta}}
\end{aligned} \tag{84}$$

Defining $\quad a_1=\tfrac{1}{2}\left(a+b\right)$ (arithmetic mean of a,b) (85)

and $\quad g_1=\sqrt{ab}$ (geometric mean of a,b) (86)

We substitute (84) into (83) and with (78) write

$$\boxed{F\left(\phi,k\right)=\tfrac{1}{2}aI\left(\beta,a_1,g_1\right)} \tag{87}$$

Defining $\phi=\phi_0,\beta=\phi_1;\ k=k_0;\ a_0=\mathrm{a},g_0=b$ and equating (79) and (87) gives

$$\boxed{F\left(\phi_0,k_0\right)=I\left(\phi_0,a_0,g_0\right)=\tfrac{1}{2}I\left(\phi_1,a_1,g_1\right)} \tag{88}$$

using the transformation $k_0^2=1-\dfrac{g_0^2}{a_0^2}$; the arithmetic and geometric means $a_1=\tfrac{1}{2}\left(a_0+g_0\right)$, $g_1=\sqrt{a_0g_0}$ respectively; and $\tan\left(\phi_1-\phi_0\right)=k_0'\tan\phi_0$ where $k_0'=\dfrac{g_0}{a_0}$

The right-hand-side in (88) can be iterated as

$$\begin{aligned}
&\tfrac{1}{2}\tfrac{1}{2}I\left(\phi_2,a_2,g_2\right) \\
&=\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}I\left(\phi_3,a_3,g_3\right)
\end{aligned}$$

giving *Landen's Descending Transformation (arithmetic-geometric mean)*

$$F\left(\phi_0, k_0\right) = \left(\tfrac{1}{2}\right)^n I\left(\phi_n, a_n, g_n\right) \tag{89}$$

with the arithmetic and geometric means from the recurrence relationships

$$a_{n+1} = \tfrac{1}{2}\left(a_n + g_n\right) \text{ and } g_{n+1} = \sqrt{a_n g_n} \quad \text{with starting values } a_0 = a, g_0 = b \tag{90}$$

and the complimentary moduli, moduli and amplitudes from

$$k'_n = \frac{g_n}{a_n};\ k_{n+1} = \frac{1 - k'_n}{1 + k'_n} \text{ and } \tan\left(\phi_{n+1} - \phi_n\right) = k'_n \tan\phi_n \quad \text{with starting values } a_0, g_0, \phi_0 \tag{91}$$

and

$$I\left(\phi_n, a_n, g_n\right) = \int_0^{\phi_n} \frac{d\theta}{\sqrt{a_n^2 \cos^2\theta + g_n^2 \sin^2\theta}} \tag{92}$$

The sequence of arithmetic and geometric means defined by (90) converge rapidly to the arithmetic-geometric mean: $a_n \to M\left(a,b\right)$ as $n \to \infty$; so with

$$\lim_{n\to\infty} I\left(\phi_n, a_n, g_n\right) = \int_0^{\hat{\phi}} \frac{d\theta}{\sqrt{M^2\cos^2\theta + M^2\sin^2\theta}} = \frac{\hat{\phi}}{M} \text{ where } M = M\left(a,b\right).$$

Since $a_n \geq M \geq g_n$, then a formula for the approximate evaluation of the elliptic integral of the first kind is (King 1924, eq. (25))

$$F\left(\phi_0, k_0\right) \approx \left(\tfrac{1}{2}\right)^n \frac{\phi_n}{a_n} \text{ or } \left(\tfrac{1}{2}\right)^n \frac{\phi_n}{g_n} \tag{93}$$

since $\hat{\phi} \approx \phi_n$

Table 7 shows numerical values for $F\left(\phi, k\right)$ (30 decimal places) using (93).

n	a_n	g_n
0	1.000000000000000000000000000000	0.996794863550168982310111529389
1	0.998397431775084491155055764694	0.998396145600617281504712946656
2	0.998396788687850886329884355675	0.998396788687643773689543697523
3	0.998396788687747330009714026599	0.998396788687747330009714021229
4	0.998396788687747330009714023914	0.998396788687747330009714023914
5	0.998396788687747330009714023914	0.998396788687747330009714023914

n	k_n	ϕ_n *(degrees)*
0	0.080000000000000000000000000000	60.000000000000000000000000000000
1	0.001605140572193028090147065969	119.920289644141316680103929766278
2	0.000000644119893905114137330012	239.840611197798147947606152992362
3	0.000000000000103722609431105390	479.681222395591132768280569471756
4	0.000000000000000000000000002690	959.362444791182265536561139076104
5	0.000000000000000000000000000000	1918.724889582364531073122278152207

Table 7. Landen's descending transformation (arithmetic-geometric mean)

Sequences $\left\{a_n\right\}, \left\{b_n\right\}, \left\{k_n\right\}$ and $\left\{\phi_n\right\}$ for $F\left(\tfrac{1}{3}\pi, 0.08\right)$

For $n = 5$ the numerical values (30 decimal places) for (93) are

$$\begin{aligned}\left(\tfrac{1}{2}\right)^5 &= 0.031250000000000000000000000000 \\ a_5 &= 0.998396788687747330009714023914 \\ \phi_5 &= 1918.724889582364531073122278152207 \text{ degrees} \\ F\left(\tfrac{1}{3}\pi, 0.08\right) &\approx 1.048182544461865455398483533571\end{aligned}$$

Table 8 shows numerical values of $F\left(\phi, k\right)$ (15 decimal places) evaluated using Maxima[2] with Landen's ascending transformation [(61)] and are identical to values from *Maple*[3] shown in Rösch (2011, Table 8).

The Maxima code is shown below the table and includes the three different ways that $F\left(\phi, k\right)$ may be evaluated: Landen's ascending transformation [(61)]; Landen's descending transformation [(75)]; and Landen's descending transformation and the arithmetic-geometric mean [(93)]

	phi = 30 deg	phi = 50 deg	phi = 70 deg	phi = 90 deg
F(phi,0.001)	0.523598798244820	0.872664721062379	1.221730701480299	1.570796719494199
F(phi,0.100)	0.523825500165390	0.873617925869649	1.223991375207876	1.574745561517356
F(phi,0.500)	0.529428627051906	0.898245235942278	1.285300585743293	1.685750354812596
F(phi,0.900)	0.543882214161571	0.974638984519665	1.535524776559492	2.280549138422770
F(phi,0.999)	0.549247510706947	1.010262233111217	1.732286917108384	4.495596395842144

Table 8. $F\left(\phi, k\right)$ evaluated using Landen's ascending transformation

Maxima code for production of Table 8

```
/***********************************************************************/
/* elint1_Table.mac
   Maxima program for calculation of a Table of elliptic integrals
   of the First Kind using Landen transformations.
*/

/* set precision for bigfloat variables*/
fpprec:36$

/* Function 1 to evaluate elliptic integral of First Kind
   using Landen's ascending transformation
*/
F1(phi,k) := block([product : 1/k, tol : 1.0b-36],
    while (1-k)>tol do block
```

[2] **Maxima** is a computer algebra system that yields high precision numerical results by using exact fractions, arbitrary precision integers, and variable precision floating point numbers. http://maxima.sourceforge.net/

[3] **Maple** is a commercial computer algebra system developed and sold commercially by Maplesoft. The current major version is version 17 which was released in March 2013. http://www.maplesoft.com/products/maple/

```
        (phi : (asin(k*sin(phi))+phi)/2,
         k : 2*sqrt(k)/(1+k),
         product : product*k),
    return(sqrt(product)*log(tan(pion4+phi/2))))$


/* Function 2 to evaluate elliptic integral of First Kind
   using Landen's descending transformation
*/
F2(phi,k) := block([product : 1.0b0, kp : sqrt(1-k*k), tol : 1.0b-36],
    while k > tol do block
       (theta : atan(kp*tan(phi)),
        phi : theta+ceiling(phi/%pi-0.5)*%pi+phi,
        k : (1-kp)/(1+kp),
        kp : sqrt(1-k*k),
        product : product*(1+k)/2),
    return(product*phi))$

/* Function 3 to evaluate elliptic integral of First Kind
   using Landen's descending transformation and the arithmetic-
   geometric mean
*/
F3(phi,k) := block([a : 1.0b0, g : sqrt(1-k*k), product : 1.0b0, tol : 1.0b-36],
    while (a-g) > tol do block
       (theta : atan(g/a*tan(phi)),
        phi : theta+ceiling(phi/%pi-0.5)*%pi+phi,
        a1 : (a+g)/2,
        g1 : sqrt(a*g),
        a : a1,
        g : g1,
        product : product*0.5),
    return(product*phi/a))$

/* set value of pi/4 */
pion4 : bfloat(%pi/4)$

/* print column headings in a Table for phi = 30 deg, 50 deg,
   70 deg and 90 deg
*/
printf(true,"~2%           ")$
for phi in [30, 50, 70, 90] do printf(true,"       ~a~2d~a","phi = ",phi," deg")$

/* evaluate F(phi,k) for different moduli k and amplitudes phi
   and print values in a table
*/
for k in [0.001b0, 0.1b0, 0.5b0, 0.9b0, 0.999b0] do block
   (printf(true,"~1% ~a~5,3h~a","F(phi,",k,")"),
    for p : 3 step 2 thru 9 do block
       (phi : bfloat(p*%pi/18),
        F : F1(phi,k),
        printf(true,"~19,15h",F)))$

printf(true,"~2%")$
```

Evaluating Elliptic Integrals of the Second Kind

An equation that will be useful in evaluating elliptic integrals of the second kind is (Rösch 2011, eq. (35))

$$E\left(\phi,k\right)+k\sin\phi=\int_0^\phi\left(\sqrt{1-k^2\sin^2\theta}+k\cos\theta\right)d\theta \tag{94}$$

Using (39) and (46) we may express the integrand of (94) as

$$\sqrt{1-k^2\sin^2\theta}+k\cos\theta=\cos\left(2\omega-\theta\right)+k\cos\theta=\sqrt{1+2k\cos 2\omega+k^2} \tag{95}$$

and substituting this result into (94) gives

$$E\left(\phi,k\right)+k\sin\phi=\int_0^\phi\sqrt{1+2k\cos 2\omega+k^2}\ d\theta \tag{96}$$

An expression for $d\theta$ is obtained by re-arranging (38) and using (95) to give

$$d\theta=\frac{2\cos\left(2\omega-\theta\right)d\omega}{\cos\left(2\omega-\theta\right)+k\cos\theta}=\frac{2\cos\left(2\omega-\theta\right)d\omega}{\sqrt{1+2k\cos 2\omega+k^2}} \tag{97}$$

With the aid of $\cos\left(A-B\right)=\cos A\cos B+\sin A\sin B$ and (42) and (43) we write the numerator of the right-hand-side of (97) as

$$\begin{aligned}2\cos\left(2\omega-\theta\right)&=2\left(\cos 2\omega\cos\theta+\sin 2\omega\sin\theta\right)\\&=2\left(\frac{\cos 2\omega\left(k+\cos 2\omega\right)}{\sqrt{1+2k\cos 2\omega+k^2}}+\frac{\sin^2 2\omega}{\sqrt{1+2k\cos 2\omega+k^2}}\right)\\&=\frac{2\left(1+k\cos 2\omega\right)}{\sqrt{1+2k\cos 2\omega+k^2}}\end{aligned} \tag{98}$$

and substituting this result into (97) gives another expression for $d\theta$

$$d\theta=\frac{2\left(1+k\cos 2\omega\right)d\omega}{1+2k\cos 2\omega+k^2} \tag{99}$$

Now substituting (99) into (96) gives (Rösch 2011, eq. (36))

$$E\left(\phi,k\right)+k\sin\phi=\int_0^\psi\frac{2\left(1+k\cos 2\omega\right)}{\sqrt{1+2k\cos 2\omega+k^2}}\ d\omega \tag{100}$$

Bearing in mind (47) and (51) we may write (100) as

$$E\left(\phi,k\right)+k\sin\phi=\int_0^\psi\frac{2\left(1+k\right)\left(1+k\cos 2\omega\right)}{\left(1+k\right)^2\sqrt{1-q^2\sin^2\omega}}\ d\omega \tag{101}$$

where $\sin\left(2\psi-\phi\right)=k\sin\phi$ and $q=\dfrac{2\sqrt{k}}{1+k}$.

The integrand on the right-hand-side of (101) can be simplified by the following sequence of manipulations

Firstly, multiplying the terms in the numerator then using $\cos 2A=1-2\sin^2 A$ gives

$$\frac{2\left(1+k\right)\left(1+k\cos 2\omega\right)}{\left(1+k\right)^2\sqrt{1-q^2\sin^2\omega}}=\frac{2+2k-4k\sin^2\omega+2k+2k^2-4k^2\sin^2\omega}{\left(1+k\right)^2\sqrt{1-q^2\sin^2\omega}}$$
$$=\frac{2\left(1+k\right)^2-4k\sin^2\omega-4k^2\sin^2\omega}{\left(1+k\right)^2\sqrt{1-q^2\sin^2\omega}}$$

Secondly, dividing numerator and denominator of the left-hand-side by $\left(1+k\right)^2$ and using (51) gives

$$\frac{2\left(1+k\right)\left(1+k\cos 2\omega\right)}{\left(1+k\right)^2\sqrt{1-q^2\sin^2\omega}}=\frac{2-\dfrac{4k}{\left(1+k\right)^2}\sin^2\omega-k\dfrac{4k}{\left(1+k\right)^2}\sin^2\omega}{\sqrt{1-q^2\sin^2\omega}}$$
$$=\frac{2-q^2\sin^2\omega-kq^2\sin^2\omega}{\sqrt{1-q^2\sin^2\omega}}$$

Thirdly, simplifying the numerator of the left-hand-side in the following sequence

$$\frac{2\left(1+k\right)\left(1+k\cos 2\omega\right)}{\left(1+k\right)^2\sqrt{1-q^2\sin^2\omega}}=\frac{1-q^2\sin^2\omega+1-k+k-kq^2\sin^2\omega}{\sqrt{1-q^2\sin^2\omega}}$$
$$=\frac{\left(1-q^2\sin^2\omega\right)+\left(1-k\right)+k\left(1-q^2\sin^2\omega\right)}{\sqrt{1-q^2\sin^2\omega}}$$
$$=\frac{\left(1+k\right)\left(1-q^2\sin^2\omega\right)+\left(1-k\right)}{\sqrt{1-q^2\sin^2\omega}}$$

Finally we have

$$\frac{2\left(1+k\right)\left(1+k\cos 2\omega\right)}{\left(1+k\right)^2\sqrt{1-q^2\sin^2\omega}}=\left(1+k\right)\sqrt{1-q^2\sin^2\omega}+\frac{1-k}{\sqrt{1-q^2\sin^2\omega}} \tag{102}$$

Substituting (102) into (101) and using (7) and (8) gives (5) stated again as

$$\boxed{E\left(\phi,k\right)+k\sin\phi=\left(1+k\right)E\left(\psi,q\right)+\left(1-k\right)F\left(\psi,q\right)} \tag{103}$$

with $\psi<\phi$ and $q>k$. The amplitudes ϕ,ψ and moduli k,q are linked by the equations:

$$\sin\left(2\psi-\phi\right)=k\sin\phi \tag{104}$$

$$q=\frac{2\sqrt{k}}{1+k} \tag{105}$$

Defining $k = k_0$, $q = k_1$ and $\phi = \phi_0$, $\psi = \phi_1$ we may write (103) as

$$E\left(\phi_0, k_0\right) = \left(1 + k_0\right) E\left(\phi_1, k_1\right) + \left(1 - k_0\right) F\left(\phi_1, k_1\right) - k_0 \sin\phi_0 \tag{106}$$

And defining

$$C_n = \left(1 - k_n\right) F\left(\phi_{n+1}, k_{n+1}\right) - k_n \sin\phi_n \tag{107}$$

gives (106) as

$$E\left(\phi_0, k_0\right) = \left(1 + k_0\right) E\left(\phi_1, k_1\right) + C_0 \tag{108}$$

and then by advancing the indices we may write

$$\begin{aligned}
&E\left(\phi_1, k_1\right) = \left(1 + k_1\right) E\left(\phi_2, k_2\right) + C_1 \\
&E\left(\phi_2, k_2\right) = \left(1 + k_2\right) E\left(\phi_3, k_3\right) + C_2 \\
&\cdots \\
&E\left(\phi_n, k_n\right) = \left(1 + k_n\right) E\left(\phi_{n+1}, k_{n+1}\right) + C_n
\end{aligned} \tag{109}$$

We may use (108) and (109) and write (106) as

$$\begin{aligned}
E\left(\phi_0, k_0\right) = &\left(1 + k_0\right)\left(1 + k_1\right)\left(1 + k_2\right)\left(1 + k_3\right)\cdots\left(1 + k_{n-1}\right)\left(1 + k_n\right) E\left(\phi_{n+1}, k_{n+1}\right) \\
&+\left(1 + k_0\right)\left(1 + k_1\right)\left(1 + k_2\right)\left(1 + k_3\right)\cdots\left(1 + k_{n-1}\right) C_n \\
&+\left(1 + k_0\right)\left(1 + k_1\right)\left(1 + k_2\right)\cdots\left(1 + k_{n-2}\right) C_{n-1} \\
&+\cdots \\
&+\left(1 + k_0\right)\left(1 + k_1\right)\left(1 + k_2\right) C_3 + \left(1 + k_0\right)\left(1 + k_1\right) C_2 + \left(1 + k_0\right) C_1 + C_0
\end{aligned} \tag{110}$$

Let $\lim\limits_{n\to\infty} \phi_n = \hat{\phi}$ and noting that $\lim\limits_{n\to\infty} k_n = 1$ (see Arithmetic Geometric Mean) we write

$$\lim_{n\to\infty} E\left(\phi_n, k_n\right) = E\left(\hat{\phi}, 1\right) = \int_0^{\hat{\phi}} \sqrt{1 - \sin^2\theta}\, d\theta = \int_0^{\hat{\phi}} \cos\theta\, d\theta = \sin\hat{\phi} \tag{111}$$

and

$$\lim_{n\to\infty} C_n = -\sin\hat{\phi} \tag{112}$$

Noting that $\lim\limits_{n\to\infty}\left(1 + k_n\right) E\left(\phi_{n+1}, k_{n+1}\right) = 2\sin\hat{\phi}$ since $\left\{E\left(\phi_n, k_n\right)\right\}$ is bounded and substituting (112) into (110) and simplifying gives

$$\boxed{\begin{aligned}
E\left(\phi_0, k_0\right) = &C_0 + \left(1 + k_0\right) C_1 + \left(1 + k_0\right)\left(1 + k_1\right) C_2 + \left(1 + k_0\right)\left(1 + k_1\right)\left(1 + k_2\right) C_3 \\
&+\cdots \\
&+\left(1 + k_0\right)\left(1 + k_1\right)\left(1 + k_2\right)\cdots\left(1 + k_{n-3}\right) C_{n-2} \\
&+\left(1 + k_0\right)\left(1 + k_1\right)\left(1 + k_2\right)\left(1 + k_3\right)\cdots\left(1 + k_{n-2}\right) C_{n-1} \\
&+\left(1 + k_0\right)\left(1 + k_1\right)\left(1 + k_2\right)\left(1 + k_3\right)\cdots\left(1 + k_{n-1}\right) \sin\hat{\phi}
\end{aligned}} \tag{113}$$

where coefficients C_n are given by (107); $F(\phi, k)$ is an elliptic integral of the first kind and the sequences of moduli $\{k_n\}$ and amplitudes $\{\phi_n\}$ are obtained from the recurrence relationships

$$k_{n+1} = \frac{2\sqrt{k_n}}{1+k_n} \quad \text{with starting value } k_0 \tag{114}$$

$$\sin\left(2\phi_{n+1} - \phi_n\right) = k_n \sin\phi_n \quad \text{with starting values } k_0, \phi_0 \tag{115}$$

For the purpose of numerical evaluation (113) may be truncated as

$$\boxed{\begin{aligned} E\left(\phi_0, k_0\right) \approx\, & C_0 + \left(1+k_0\right)C_1 + \left(1+k_0\right)\left(1+k_1\right)C_2 + \left(1+k_0\right)\left(1+k_1\right)\left(1+k_2\right)C_3 \\ & +\cdots \\ & +\left(1+k_0\right)\left(1+k_1\right)\left(1+k_2\right)\cdots\left(1+k_{n-3}\right)C_{n-2} \\ & +\left(1+k_0\right)\left(1+k_1\right)\left(1+k_2\right)\left(1+k_3\right)\cdots\left(1+k_{n-2}\right)C_{n-1} \\ & +\left(1+k_0\right)\left(1+k_1\right)\left(1+k_2\right)\left(1+k_3\right)\cdots\left(1+k_{n-1}\right)\sin\phi_n \end{aligned}} \tag{116}$$

Noting (26) – (30) we may write here; with $a_0 = 1, g_0 = k_0$ and the arithmetic and geometric means defined as $a_{n+1} = \frac{1}{2}\left(a_n + g_n\right)$ and $g_{n+1} = \sqrt{a_n g_n}$ respectively

$$k_{n+1} = \frac{g_{n+1}}{a_{n+1}} = \frac{2\sqrt{k_n}}{1+k_n} \tag{117}$$

$$1 + k_n = 2\frac{a_{n+1}}{a_n} \tag{118}$$

and
$$\left(1+k_0\right)\left(1+k_1\right)\left(1+k_2\right)\cdots\left(1+k_{n-1}\right) = 2\frac{a_1}{a_0}\cdot 2\frac{a_2}{a_1}\cdot 2\frac{a_3}{a_2}\cdots 2\frac{a_n}{a_{n-1}} = 2^n a_n \tag{119}$$

Using (119) in (116) gives an alternative expression

$$\boxed{\begin{aligned} E\left(\phi_0, k_0\right) \approx\, & C_0 + 2a_1C_1 + 2^2a_2C_2 + 2^3a_3C_3 + \cdots \\ & + 2^{n-2}a_{n-2}C_{n-2} + 2^{n-1}a_{n-1}C_{n-1} + 2^n a_n \sin\phi_n \end{aligned}} \tag{120}$$

Equation (120) can be simplified further by considering $C_0, C_1, C_2, \ldots$ given by (107) that contains $F\left(\phi_{n+1}, k_{n+1}\right)$ that can, by using (56), be written as

$$F\left(\phi_{n+1}, k_{n+1}\right) = \tfrac{1}{2}\left(1+k_0\right)\tfrac{1}{2}\left(1+k_1\right)\tfrac{1}{2}\left(1+k_2\right)\cdots\tfrac{1}{2}\left(1+k_n\right)F\left(\phi_0, k_0\right) \tag{121}$$

Using (119) we may write (121) as

$$F\left(\phi_{n+1}, k_{n+1}\right) = a_{n+1}F\left(\phi_0, k_0\right) \tag{122}$$

Also, using $k_n = \dfrac{g_n}{a_n}$ in (107) we may write

$$1 - k_n = \frac{a_n - g_n}{a_n} = 2\frac{c_{n+1}}{a_n} \quad \text{and} \quad k_n \sin\phi_n = \frac{g_n}{a_n}\sin\phi_n \tag{123}$$

Now using (122) and (123) in (107) gives

$$C_n = 2\frac{c_{n+1}a_{n+1}}{a_n}F\left(\phi_0,k_0\right) - \frac{g_n}{a_n}\sin\phi_n \tag{124}$$

where $a_{n+1} = \frac{1}{2}\left(a_n + g_n\right)$ and $g_{n+1} = \sqrt{a_n g_n}$ are arithmetic and geometric means respectively and $c_{n+1} = \frac{1}{2}\left(a_n - g_n\right)$, and substituting (124) into (120) gives

$$\boxed{\begin{aligned} E\left(\phi_0,k_0\right) \approx{}& F\left(\phi_0,k_0\right)\left\{2a_1c_1 + 2^2a_2c_2 + 2^3a_3c_3 + \cdots + 2^n a_n c_n\right\} \\ &-\left\{g_0\sin\phi_0 + 2g_1\sin\phi_1 + 2^2g_2\sin\phi_2 + \cdots + 2^{n-1}g_{n-1}\sin\phi_{n-1}\right\} + 2^n a_n \sin\phi_n \end{aligned}} \tag{125}$$

Table 9 shows numerical values of $E\left(\phi,k\right)$ (15 decimal places) evaluated using Maxima with Landen's ascending transformation and the arithmetic-geometric mean [(125)] and are identical with those shown in Rösch (2011, Table 13) evaluated using Maple.

The Maxima code is shown below the table and includes the function to evaluate $F\left(\phi,k\right)$ using Landen's ascending transformation [(61)].

	phi = 30 deg	phi = 50 deg	phi = 70 deg	phi = 90 deg
E(phi,0.001)	0.523598752951780	0.872664530931969	1.221730251311829	1.570795934095741
E(phi,0.500)	0.517881934859938	0.848316628033472	1.163176859928730	1.467462209339427
E(phi,0.999)	0.500049276809973	0.766288871196247	0.940486775266712	1.003994409965508

Table 9. $E\left(\phi,k\right)$ evaluated using Landen's ascending transformation

Maxima code for production of Table 9

```
/*******************************************************************/
/* elint2_Table.mac
   Maxima program for calculation of a Table elliptic integrals
   of the Second Kind using Landen transformations.
*/

/* set precision for bigfloat variables*/
fpprec:36$

/* Function to evaluate elliptic integral of First Kind
   using Landen's ascending transformation
*/
F(phi,k) := block([product : 1/k, tol : 1.0b-36],
    while (1-k)>tol do block
       (phi : (asin(k*sin(phi))+phi)/2,
        k : 2*sqrt(k)/(1+k),
        product : product*k),
```

```
    return(sqrt(product)*log(tan(pion4+phi/2))))$

/* Function to evaluate an elliptic integral of the Second Kind
   using Landen's ascending transformation and the arithmetic-
   geometric mean
*/
E(phi,k) := block([a : 1.0b0, g : k, n : 0, sum1 : 0.0b0, sum2 : 0.0b0, tol : 1.0b-36],
    F : F(phi,k),
    while (a-g) > tol do block
       (sum2 : sum2 + (2^n)*g*sin(phi),
        a1 : (a + g)/2,
        g1 : sqrt(a*g),
        c1 : (a - g)/2,
        phi : (asin(k*sin(phi))+phi)/2,
        sum1 : sum1 + (2^(n+1))*a1*c1,
        a : a1,
        g : g1,
        k : g1/a1,
        n : n+1),
    return(F*sum1-sum2+(2^n)*a*sin(phi)))$

/* set value of pi/4 */
pion4 : bfloat(%pi/4)$

/* print column headings in a Table for phi = 30 deg, 50 deg,
   70 deg and 90 deg
*/
printf(true,"~2%           ")$
for phi in [30, 50, 70, 90] do printf(true,"       ~a~2d~a","phi = ",phi," deg")$

/* evaluate E(phi,k) for different moduli k and amplitudes phi
   and print values in a table
*/
for k in [0.001b0, 0.5b0, 0.999b0] do block
   (printf(true,"~1% ~a~5,3h~a","E(phi,",k,")"),
    for p : 3 step 2 thru 9 do block
       (phi : bfloat(p*%pi/18),
        E : E(phi,k),
        printf(true,"~19,15h",E)))$

printf(true,"~2%")$
```

Meridian distance on an ellipsoid - an application of Elliptic Integrals

In Geodesy, meridian distance M on an ellipsoid of revolution is the distance along a meridian from the equator to the point having latitude ϕ. An ellipsoid whose semi-axes lengths are a and b and $a > b$ has the following geometric constants: flattening f; eccentricity e; 2nd eccentricity e'; 3rd flattening n and polar radius of curvature c defined as

$$f = \frac{a-b}{a}; \quad e = \sqrt{\frac{a^2-b^2}{a^2}}; \quad e' = \sqrt{\frac{a^2-b^2}{b^2}}; \quad n = \frac{a-b}{a+b}; \quad c = \frac{a^2}{b} \tag{126}$$

and inter-related as follows

$$e^2 = f\left(2-f\right); \quad e'^2 = \frac{e^2}{1-e^2}; \quad n = \frac{f}{2-f}; \quad c = a\left(\frac{1+n}{1-n}\right) \tag{127}$$

As an example, the Geodetic Reference System 1980 (GRS80) adopted by the XVII General Assembly of the International Union of Geodesy and Geophysics (IUGG) in Canberra, December 1979 has a reference ellipsoid with the following geometric parameters:

semi-major axis $a = 6378137$ metres and flattening $f = 1/298.257222101$

Computed constants for the GRS80 ellipsoid are:

$$b = a(1-f) = 6\,356\,752.314 \text{ metres} \qquad c = \frac{a}{1-f} = 6\,399\,593.626 \text{ metres}$$

$$e^2 = f(2-f) = 6.694\,380\,023\text{e-}003 \qquad e'^2 = \frac{f(2-f)}{(1-f)^2} = 6.739\,496\,775\text{e-}003$$

$$n = \frac{f}{2-f} = 1.679\,220\,395\text{e-}003$$

Meridian arc length is defined by the differential relationship

$$dM = \rho\, d\phi \tag{128}$$

where ρ is the radius of curvature in the meridian plane and given by

$$\rho = \frac{a(1-e^2)}{(1-e^2\sin^2\phi)^{3/2}} = \frac{a(1-e^2)}{W^3} \tag{129}$$

Alternatively, the radius of curvature in the meridian plane is also given by

$$\rho = \frac{a^2}{b(1+e'^2\cos^2\phi)^{3/2}} = \frac{c}{V^3} \tag{130}$$

where the latitude functions V and W are defined as

$$W^2 = 1 - e^2\sin^2\phi; \qquad V^2 = 1 + e'^2\cos^2\phi = \frac{1 + n^2 + 2n\cos 2\phi}{(1-n)^2} \tag{131}$$

Substituting (129) into (128) leads to series formula for the meridian distance M as a function of latitude ϕ and powers of e^2. Substituting (130) into (128) leads to series formula for M as a function of ϕ and powers of n.

Series formula involving powers of e^2 are more commonly found in the geodetic literature but, series formula involving powers of n are more compact requiring fewer terms for the same numerical accuracy since $n \approx \frac{1}{4}e^2$. Following Helmert's[4] method of development (Deakin & Hunter 2013) and with some algebra we may write

$$M = \int_0^\phi \frac{c}{V^3}\, d\theta = \frac{a}{1+n}\int_0^\phi \left(1-n^2\right)^2 \left(1+n^2+2n\cos 2\theta\right)^{-3/2} d\theta \tag{132}$$

The integrand in (132) is expanded by use of Taylor series followed by term-by-term integration and simplification. This process yields the usual 'series form' expression for meridian distances. Maxima can be used to evaluate the integral and M can be written as

$$M = \frac{a}{1+n}\left\{\begin{array}{l} c_0\phi + c_2\sin 2\phi + c_4 \sin 4\phi \\ \quad +c_6\sin 6\phi + c_8\sin 8\phi \\ \quad +c_{10}\sin 10\phi + c_{12}\sin 12\phi \\ \quad +c_{14}\sin 14\phi + c_{16}\sin 16\phi + \cdots \end{array}\right\} \tag{133}$$

where the coefficients $\{c_n\}$ are to order n^8 as follows

$$\begin{array}{ll}
c_0 = 1 + \frac{1}{4}n^2 + \frac{1}{64}n^4 + \frac{1}{256}n^6 + \frac{25}{16384}n^8 \cdots & c_2 = -\frac{3}{2}n + \frac{3}{16}n^3 + \frac{3}{128}n^5 + \frac{15}{2048}n^7 + \cdots \\
c_4 = \frac{15}{16}n^2 - \frac{15}{64}n^4 - \frac{75}{2048}n^6 - \frac{105}{8192}n^8 - \cdots & c_6 = -\frac{35}{48}n^3 + \frac{175}{768}n^5 + \frac{245}{6144}n^7 + \cdots \\
c_8 = \frac{315}{512}n^4 - \frac{441}{2048}n^6 - \frac{1323}{32768}n^8 - \cdots & c_{10} = -\frac{693}{1280}n^5 + \frac{2079}{10240}n^7 + \cdots \\
c_{12} = \frac{1001}{2048}n^6 - \frac{1573}{8192}n^8 - \cdots & c_{14} = -\frac{6435}{14336}n^7 + \cdots \\
c_{16} = \frac{109395}{262144}n^8 - \cdots &
\end{array} \tag{134}$$

For the GRS80 ellipsoid, the meridian distance M for $\phi = 60^\circ$ evaluated using (133) is

$$M = 6654072.81936745\,\text{metres}$$

An alternative to this usual method of development can be developed by substituting (129) into (128) and writing

[4] German geodesist F.R. Helmert who wrote *Die mathematischen und physikalischen Theorieen der höheren Geodäsie* (Part I was published in 1880 and Part II in 1884) that laid the foundations of modern geodesy. (Part I is devoted to the mathematical aspects of geodesy.) In 1876 he discovered the chi-squared distribution as the distribution of the sample variance for a normal distribution. This discovery and other of his work was described in German textbooks, including his own, but was unknown in English, and hence later rediscovered by English statisticians – the chi-squared distribution by Karl Pearson (1900), and the application to the sample variance by 'Student' and Fisher.

$$M = \int_0^{\phi} \frac{a\left(1-e^2\right)}{W^3} d\theta = a\left(1-e^2\right)\int_0^{\phi} \frac{d\theta}{\left(\sqrt{1-e^2\sin^2\theta}\right)^3} \tag{135}$$

With $e \equiv k$ and using (6) and (11) we may write (135) as

$$M = a\left(1-e^2\right)\int_0^{\phi} \frac{d\theta}{\Delta^3} = a\left(1-e^2\right) = a\left(1-e^2\right)\Pi\left(\phi, e\right) \tag{136}$$

And using (14) we write

$$M = a\left\{E\left(\phi, e\right) - \frac{e^2 \sin\phi\cos\phi}{\sqrt{1-e^2\sin^2\phi}}\right\} \tag{137}$$

where $E\left(\phi, e\right)$ is the elliptic integral of the Second Kind that may be evaluated using (125).

For the GRS80 ellipsoid, the meridian distance M for $\phi = 60^\circ$ evaluated (to 30 decimal places) using Maxima and (137) and (125) is

$$M = 6654072.819367444406819108934413675127 \text{ metres}$$

The Maxima program *mdist_ elint2.mac* is shown below followed by the program output in Table 10.

```
/**********************************************************************/
/* mdist_elint2.mac
   Maxima program for calculation of meridian distance using an
   elliptic integral of the Second Kind using Landen transformations.
*/

/* set precision for bigfloat variables*/
fpprec:48$

/* Function to evaluate elliptic integral of First Kind
   using Landen's ascending transformation
*/
F(phi,k) := block([product : 1/k, tol : 1.0b-36],
    while (1-k)>tol do block
       (phi : (asin(k*sin(phi))+phi)/2,
        k : 2*sqrt(k)/(1+k),
        product : product*k),
    return(sqrt(product)*log(tan(pion4+phi/2))))$

/* Function to evaluate an elliptic integral of the Second Kind
   using Landen's ascending transformation and the arithmetic-
   geometric mean
*/
E(phi,k) := block([a : 1.0b0, g : k, n : 0, sum1 : 0.0b0, sum2 : 0.0b0, tol : 1.0b-40],
    F : F(phi,k),
    while (a-g) > tol do block
       (sum2 : sum2 + (2^n)*g*sin(phi),
        a1 : (a + g)/2,
        g1 : sqrt(a*g),
```

```
        c1 : (a - g)/2,
        phi : (asin(k*sin(phi))+phi)/2,
        sum1 : sum1 + (2^(n+1))*a1*c1,
        a : a1,
        g : g1,
        k : g1/a1,
        n : n+1),
    return(F*sum1-sum2+(2^n)*a*sin(phi)))$

/* set degree to radian conversion */
d2r : bfloat(180/%pi)$

/* set value of pi/4 */
pion4 : bfloat(%pi/4)$

/* set GRS80 ellipsoid parameters a and f */
a : 6378137.0b0$
flat : 298.257222101b0$
f : 1/flat$

/* computed ellipsoid constants */
k2 : f*(2-f)$
k : sqrt(k2)$

/* set latitude */
phi : bfloat(%pi/3)$
/* phi : bfloat(%pi/2)$ */

/* compute meridian distance */
E : E(phi,k)$
mdist : a*(E-k2*sin(phi)*cos(phi)/sqrt(1-k2*sin(phi)^2))$

/* print results */
printf(true,"~1% ~a~38,30h~a","       a = ",a," metres")$
printf(true,"~1% ~a~38,30h","    flat = ",flat)$
printf(true,"~1% ~a~38,30h","       f = ",f)$
printf(true,"~1% ~a~38,30h~a","     phi = ",phi*d2r,"(degrees)")$
printf(true,"~1% ~a~38,30h","       e = ",k)$
printf(true,"~1% ~a~38,30h","E(phi,e) = ",E)$
printf(true,"~1% ~a~38,30h~a","   mdist = ",mdist," metres")$

printf(true,"~2%")$
```

Output from program *mdist_ elint2.mac*

```
       a = 6378137.000000000000000000000000000000 metres
    flat =     298.257222101000000000000000000000
       f =       0.003352810681182318935434146126
     phi =      60.000000000000000000000000000000(degrees)
       e =       0.081819191042815790145895739896
E(phi,e) =       1.046168817527900319688127443142
   mdist = 6654072.819367444406819108934413675127 metres
```

Table 10. Meridian distance evaluated using the elliptic integral $E\left(\phi, e\right)$ and (137)

As a way of confirming the method of computation, program *mdist_ elint2.mac* was modified to produce a table of meridian distances M for latitudes 30°, 45°, 60° and 90° on Bessel's ellipsoid ($a = 6377397.155\,\mathrm{m}$, $e = 0.08169683121517$) giving

```
      a =  6377397.155000000000000 metres
   flat =      299.152812853972934
      f =        0.003342773181572
      e =        0.081696831215170

             phi = 30 deg           phi = 45 deg           phi = 60 deg           phi = 90 deg
mdist =   3319786.509543301836   10000855.764435535539   6653376.120611621107   10000855.764435535539
```

These results agree with those given in Dorrer (1999, p. 97).

An alternative formula for computing meridian distance using elliptic integrals can be derived from the differential arc length of the meridian ellipse whose Cartesian equation is

$$\frac{w^2}{a^2} + \frac{z^2}{b^2} = 1 \tag{138}$$

where the w-axis lies in the equatorial plane of the ellipsoid and the z-axis is coincident with the axis of revolution of the ellipsoid; and a point on the meridian ellipse has coordinates

$$\begin{aligned} w &= a\cos\psi \\ z &= b\sin\psi \end{aligned} \tag{139}$$

where ψ is the parametric (or reduced) latitude and

$$\tan\psi = \frac{b}{a}\tan\phi = \sqrt{1-e^2}\,\tan\phi = \left(1-f\right)\tan\phi \tag{140}$$

The differential arc length ds satisfies the relationship $ds^2 = dw^2 + dz^2$ where $dw = -a\sin\psi\,d\psi$, $dz = b\cos\psi\,d\psi$ and

$$\begin{aligned} ds^2 &= \left(a^2\sin^2\psi + b^2\cos^2\psi\right)d\psi^2 \\ &= \left(a^2\left(1-\cos^2\psi\right) + b^2\cos^2\psi\right)d\psi^2 \\ &= \left(a^2 - \left(a^2-b^2\right)\cos^2\psi\right)d\psi^2 \\ &= a^2\left(1-e^2\cos^2\psi\right)d\psi^2 \end{aligned} \tag{141}$$

And the meridian distance from the equator to parametric latitude ψ is

$$M = a\int_0^{\psi}\sqrt{1-e^2\cos^2\theta}\,d\theta \tag{142}$$

This equation is not in a suitable form, but using $\beta = 90^\circ - \psi$ with $\cos^2\psi = \sin^2\beta$ in (141) gives an equation for the distance from the pole as

$$M_{POLE} = a\int_0^\beta \sqrt{1-e^2\sin^2\theta}\,d\theta = a\,E\left(\beta,e\right) \tag{143}$$

This equation was given by Legendre (1811, p. 179).

Defining Q as the quadrant distance (meridian distance from equator to pole) and noting (10) and (143) we write

$$M = Q - M_{\mathrm{POLE}} = a\left\{E\left(\tfrac{1}{2}\pi,e\right) - E\left(\beta,e\right)\right\} \tag{144}$$

where $\tan\beta = \dfrac{1}{\left(1-f\right)\tan\phi}$

Maxima program *mdist_ elint2.mac* was modified to give the following results for M for $\phi = 60^\circ$ evaluated (to 30 decimal places) using (144) and (125)

$$\begin{aligned} M_{\mathrm{POLE}} &= 3347892.909863019284699224985055609762 \text{ metres} \\ Q &= 10001965.729230463691518333919469284889 \text{ metres} \\ M &= 6654072.819367444406819108934413675127 \text{ metres} \end{aligned}$$

Output from modified program *mdist_ elint2.mac* is shown in Table 11.

```
          a =  6378137.000000000000000000000000000000 metres
       flat =      298.257222101000000000000000000000
          f =        0.003352810681182318935434146126
        phi =       60.000000000000000000000000000000(degrees)
        psi =       59.916607796611328125061711133111(degrees)
       beta =       30.083392203388671874938288866889(degrees)
          e =        0.081819191042815790145895739896
 E(beta,e) =        0.524901379487931865480347158591
     M_pole =  3347892.909863019284699224985055609762 metres
          Q = 10001965.729230463691518333919469284889 metres
      mdist =  6654072.819367444406819108934413675127 metres
```

Table 11. Meridian distance evaluated using the elliptic integral $E\left(\beta,e\right)$ and (144)

Geodesic Arc Length – another application of Meridian Distance and Elliptic Integrals

In geodesy, the *geodesic* is a unique curve on the surface of an ellipsoid defining the shortest distance between two points. It has a characteristic equation

$$\nu\cos\phi\sin\alpha = a\cos\psi\sin\alpha = C \tag{145}$$

where $\nu = c/V$ is the radius of curvature in the prime vertical plane, ϕ is latitude, α is azimuth, a is the semi-major axis of the ellipsoid, ψ is parametric latitude and C is a constant. Equation (145) is known as Clairaut's equation[5]. A geodesic oscillates over the surface of the ellipsoid between two parallels of latitude having a maximum in the northern and southern hemispheres and crossing the equator at nodes; but due to the eccentricity of the ellipsoid the geodesic will not repeat after a complete cycle. $\left|\sin\alpha\right|$ is a maximum of 1 when $\alpha = 90^\circ$ and 270° and this point is known as the geodesic vertex having latitudes ϕ_0 and ψ_0 and

$$a\cos\psi_0 = a\cos\psi\sin\alpha = C \quad \text{and} \quad \cos\psi_0 = \cos\psi\sin\alpha \tag{146}$$

The geodesic crosses the equator $\left(\phi = \psi = 0^\circ\right)$ at a node and its azimuth at this point is denoted as α_E and using (146) we obtain

$$\cos\psi_0 = \sin\alpha_E \quad \text{and} \quad \sin^2\psi_0 = \cos^2\alpha_E \tag{147}$$

Equations relating to geodesics on the ellipsoid are often developed with the aid of an *auxiliary sphere* where points on the ellipsoid having geodetic coordinates ϕ,λ (latitude and longitude respectively) correspond to points on the auxiliary sphere having coordinates ψ,ω (spherical latitude and spherical longitude respectively), noting that parametric latitude $\psi = \arctan\left(\left(1-f\right)\tan\phi\right)$ on the ellipsoid is identical to spherical latitude ψ on the auxiliary sphere. The geodesic on the ellipsoid is mapped to the auxiliary sphere as a great circle and the azimuth α of a geodesic at P on the ellipsoid is identical to the azimuth A of the great circle passing through the corresponding point P' on the auxiliary sphere since both have identical values of ψ. This is a consequence of Clairaut's equation (145). A geodesic arc of length s on the ellipsoid corresponds to a great circle arc of length σ on the auxiliary sphere and relationships between geodetic quantities $\left(\phi,\psi,\lambda,\alpha,s\right)$ and spherical quantities $\left(\psi,\omega,A=\alpha,\sigma\right)$ can be established using both spherical trigonometry and ratios of differential quantities on both surfaces. Legendre (1811, § 127, pp. 179-180) used these relationships and gave (in our notation) two differential equations

---

[5] Named in honour of the French mathematical physicist Alexis-Claude Clairaut (1713-1765) who published this result in 1733 in a paper titled *Détermination geometric de la perpendicular à la méridienne,* that was a study of geodesics on quadrics of revolution.

$$
\begin{aligned}
ds &= \sqrt{b^2 + a^2e^2 \sin^2 \psi_0 \cos^2 \sigma}\, d\sigma \\
d\omega &= \frac{\cos \psi_0}{a} \frac{\sqrt{b^2 + a^2e^2 \sin^2 \psi_0 \cos^2 \sigma}}{1 - \sin^2 \psi_0 \cos^2 \sigma} d\sigma
\end{aligned} \tag{148}
$$

where σ is a spherical arc length measured from the vertex on the auxiliary sphere.

We are interested here in geodesic arc length s, and taking the first member of (148) we write

$$
\begin{aligned}
ds^2 &= \left(b^2 + a^2e^2 \sin^2 \psi_0 \cos^2 \sigma\right) d\sigma^2 \\
&= \left(b^2 + a^2e^2 \sin^2 \psi_0 \left(1 - \sin^2 \sigma\right)\right) d\sigma^2 \\
&= \left(b^2 + a^2e^2 \sin^2 \psi_0 - a^2e^2 \sin^2 \psi_0 \sin^2 \sigma\right) d\sigma^2 \\
&= b^2 + a^2e^2 \sin^2 \psi_0 \left(1 - \frac{a^2e^2 \sin^2 \psi_0}{b^2 + a^2e^2 \sin^2 \psi_0} \sin^2 \sigma\right) d\sigma^2
\end{aligned} \tag{149}
$$

Let
$$
\varepsilon^2 = \frac{a^2e^2 \sin^2 \psi_0}{b^2 + a^2e^2 \sin^2 \psi_0} = 1 - \frac{b^2}{b^2 + a^2e^2 \sin^2 \psi_0} \tag{150}
$$

then
$$
ds = \frac{b}{\sqrt{1 - \varepsilon^2}} \left(1 - \varepsilon^2 \sin^2 \sigma\right) d\sigma \tag{151}
$$

Here, ε is the eccentricity of a 'new' ellipsoid whose semi-minor axis is b and semi-major axis is

$$
a^* = \sqrt{b^2 + a^2e^2 \sin^2 \psi_0} = \frac{b}{\sqrt{1 - \varepsilon^2}} \tag{152}
$$

and
$$
s = a^* \int_0^\sigma 1 - \varepsilon^2 \sin^2 \sigma' \; d\sigma' = a^* E\left(\sigma, \varepsilon\right) \tag{153}
$$

Comparing (153) with (143) we see that the geodesic arc s measured from the vertex is equivalent to the meridian distance from the pole of an ellipsoid a^*, b $\left(a^* > b\right)$ to a point having parametric latitude $\psi = 90^\circ - \sigma$ or $\sigma = \beta = 90^\circ - \psi$.

We may confirm this by first computing a reference geodesic arc on an ellipsoid and then comparing this with results from (153). Our reference geodesic arc is computed using 'on-line' software available at *Geographiclib – direct and inverse geodesic calculations* (http://geographiclib.sourceforge.net/cgi-bin/GeodSolve) written by C.F.F. Karney. This software is capable of nanometre accuracy for any geodesic on the surface of the ellipsoid and the algorithms are described in *Algorithms for Geodesics* (Karney 2013).

Reference geodesic arc

GRS80 ellipsoid $\left(a = 6378137 \text{ metres, } f = 1/298.257222101\right)$

azimuth at equator (node): $\alpha_E = \frac{1}{6}\pi = 30^\circ$

geodesic arc, equator to vertex: $s_0 = 9997769.059919197$ metres

arc length on auxiliary sphere, equator to vertex: $\sigma_0 = 90.00000000000000$ degrees

parametric latitude of vertex: $\psi_0 = \arccos\left(\sin\alpha_E\right) = \frac{1}{3}\pi = 60^\circ$

latitude of vertex: $\phi_0 = \arctan\left(\dfrac{\tan\psi_0}{1-f}\right) = 60.08325228717234$ degrees.

longitude difference between node and vertex: $\Delta\lambda = 89.84921850746352$ degrees

geodesic arc, equator to *P:* $s_1 = 4994873.220573560$ metres

arc length on auxiliary sphere, equator to *P:* $\sigma_1 = 45.00000000000000$ degrees

latitude *P:* $\phi_P = 37.85444005133714$ degrees

longitude difference between node and *P:* $\Delta\lambda_P = 26.48963021420673$ degrees

The Maxima function *mdist_ elint2.mac* was modified to produce the output in Table 12 for a point P on a geodesic on the GRS80 ellipsoid. The geodesic crosses the equator at an azimuth $\alpha_E = \frac{1}{6}\pi = 30^\circ$ and P is located on this geodesic such that its arc length on the auxiliary sphere from the vertex is $\sigma = \frac{1}{4}\pi = 45^\circ$. The modified function uses (153) with the new ellipsoid a^*, b with eccentricity ε from (150) and (152)

```
              a =  6378137.0000000000000000000000000000 metres
           flat =      298.2572221010000000000000000000
              f =        0.003352810681182318935434146126
              e =        0.081819191042815790145895739896
             e' =        0.082094438151917199403251490213
              b =  6356752.314140355847852106861529533079
              Q = 10001965.729230463691518333919469284889 metres
        alpha_E =       30.000000000000000000000000000000(degrees)
          psi_0 =       60.000000000000000000000000000000(degrees)
          phi_0 =       60.083252287172337202234022676318(degrees)

          a_new =  6372797.555933260801146114967789232643 metres
       flat_new =      397.176785378905828888485190217561
          f_new =        0.002517770516335696900202345472
          e_new =        0.070916865866297732761886275874
      b_new = b =  6356752.314140355847852106861529533079
          sigma =       45.000000000000000000000000000000(degrees)
E(sigma,e_new) =         0.785039191255619666532078660357
        mdist_P =  5002895.839345636695447863997672795087 metres
          Q_new =  9997769.059919197098224444256759521859 metres
        mdist_E =  4994873.220573560402776580259086726772 metres
```

Table 12. Meridian distances on 'new' ellipsoid equal to Geodesic arcs evaluated using elliptic integral $E\left(\sigma, e\right)$ and (153)

From Table 12, meridian distances `Q_new`, `mdist_P` and `mdist_E` on the new ellipsoid are equal to s_0, s_1, and $s_0 - s_1$ respectively on the reference geodesic at nanometre level. This would appear to confirm the method of evaluation using the elliptic integral of the second kind $E(\sigma, e)$. [Interestingly, the semi-major axis a^* (`a_new`) for the new ellipsoid is identical to the *reduced length*[6] m for the geodesic arc from equator to vertex. Karney (2013) makes use of the reduced length in the solution of the geodesic *inverse* problem.]

Conclusions

This paper presents methods of evaluation for elliptic integrals of the first and second kind. These methods employ Landen's transformation in three forms; ascending, descending and descending with arithmetic-geometric mean. In explaining these methods we have provided a detailed treatment of the arithmetic-geometric mean iteration that lies at the heart of elliptic integral evaluations as well as a trigonometric development of Landen's transformations. We have also provided numerical examples of the iterative schemes and hopefully demonstrated their rapid convergence to acceptable results. Indeed, the accuracy and speed of these methods is limited only by the computer 'architecture'. And they are very simple to program. To demonstrate this point, Maxima code has been given that includes two functions `F(phi,k)` and `E(phi,k)` that evaluate elliptic integrals of the first and second kind respectively. These functions should be easily translated into other computer languages.

This paper also gives two examples of how `F(phi,k)` and `E(phi,k)` could be used in place of 'conventional' geodetic formulae. These formulae are in fact truncated power series arising from alternative methods of evaluating elliptic integrals. They were appropriate when they were originally developed (early 19th century) but in this age of the computer they could be viewed as an inferior method.

Recently, papers and reports by C.F.F. Karney (2011a, 2011b and 2013) have provided algorithms for computing geodesics and transverse Mercator projection coordinates to nanometre precision. These algorithms can be regarded as the current 'gold standard'. Karney's work and the GeographicLib software library, provide alternative formulations in terms of elliptic integrals that are often regarded as 'exact' solutions and a means of

[6] The reduced length was introduced by Gauss (1902) and was also the subject of a study by the German mathematician Christoffel (1829–1900) who coined the term (Karney 2013).

comparing the accuracy of algorithms. Perhaps in the future, more geodetic software will incorporate these numerical routines.

References

Cox, D.A., 1984, 'The Arithemtic-Geometric Mean of Gauss', *L'Enseignement Mathématique,* t. 30 (1984), pp. 275-330.
Also in: Berggren, L., Borwein, J. and Borwein, P., 1997, *Pi: A Source Book,* Springer, New York, pp. 481-536.

Deakin, R.E. and Hunter, M.N., 2013, *Geometric Geodesy - Part A,* Lecture Notes, School of Mathematical and Geospatial Sciences, RMIT University, Melbourne, Australia.

DLMF, *NIST Digital Library of Mathematical Functions,* National Institute of Standards and Technology (NIST), US Department of Commerce.
http://dlmf.nist.gov/

Dorrer, E., 1999, 'From Elliptic Arc Length to Gauss-Krüger Coordinates by Analytical Continuation', *Quo vadis geodesia?* Anniversarty Fetschrift dedicated to Erik W. Grafarend, Schriftenreihe des Studiengang Geodäsie & Geoinformatik, Nr 6, Stuttgart, 1999, 9 pages.

Finch, S.R., 2003, *Mathematical Constants,* Cambridge University Press.

Gauss, C.F., 1818, 'Determinatio Attractionis quam in Punctum quodvis Positionis datae exerceret Planeta, etc.,', (Determination of the attraction which a planet exerts ...) *Comm. Gott. Soc. Reg. Scient.* iv. 1818; *Ges. Werke* (Gottingen, 1866).

Gauss, C.F., 1902, *General Investigations of Curved Surfaces of 1827 and 1825,* Princeton University Library, translation of *Disquisitiones generales circa superficies curvas,* by J.C. Morehead and A.M. Hiltebeitel.

GeographicLib, [software routines for performing conversions between geographic, UTM, UPS, MGRS, geocentric, and local cartesian coordinates, for gravity, geoid height, and geomagnetic field calculations, and for solving geodesic problems]
http ://geographiclib.sourceforge.net/

Karney, C.F.F., 2011a, 'Geodesics on an ellipsoid of revolution', Technical Report, SRI International, 29 pages, February 2011. E-print: arXiv:1102.1215v1.

Karney, C.F.F., 2011b, ‘Transverse Mercator with an accuracy of a few nanometers’, *Journal of Geodesy,* Vol. 85, Issue 8, pp. 475-485, August 2011. DOI: 10.1007/s00190-011-0445-3.

Karney, C.F.F., 2013, ‘Algorithms for geodesics’, *Journal of Geodesy,* Vol. 87, Issue 1, pp. 43-55, January 2013. DOI: 10.1007/s00190-012-0578-z; addenda: geod-addenda.html.

King, L.V., 1924, *On the Direct Numerical Calculation of Elliptic Functions and Integrals,* Cambridge University Press, London.

Jameson, G.J.O., ‘Elliptic integrals, the arithmetic-geometric mean and the Brent-Salamin algorithm for π’, Department of Mathematics and Statistics, Lancaster University, UK, 36 pages
www.maths.lancs.ac.uk/~jameson/ellagm.pdf (accessed 12-Jan-2014)

Landen, J., 1775, ‘An Investigation of a General Theorem for Finding the Length of Any Arc of Any Conic Hyperbola, by means of Two Elliptic Arcs, with Some Other New and Useful Theorems Deduced Therefrom’, *Philosophical Transactions of the Royal Society,* Vol. LXV, pp .283-289.

Legendre, A.M., 1786, ‘Mémoire sur les intégrations par arcs d’ellipse’ and ‘Second mémoire, etc.,’ *Mém. de l’Academy des Sciences de Paris,* pp. 616-643 and 644-683, Paris, 1788

Legendre, A.M., 1811, *Exercises de Calcul Integral sur Divers Ordres de Trancendantes et sur les Quadratures,* Vol. 1 [Exercises in Integral Calculus], Courcier, Paris, 1811
https://openlibrary.org/books/OL7036722M/Exercices_de_calcul_inte´gral_sur_divers_ordres_de_transcendantes_et_sur_les_quadratures (accessed 06-Feb-2014)

Legendre, A.M., 1825, ‘Traité des fonctions elliptiques et des integrals eulériennes’, Tome Premier, Imprimerie de Huzard-Courcier, Paris, 1825.

Rösch, N., 2011, ‘The derivation of algorithms to compute elliptic integrals of the first and second kind by Landen-transformation’, Boletim de Ciências Geodésicas (on-line version) Vol. 17, No.1, Curitiba, Jan/Mar 2011, p. 3-22.
http://dx.doi.org/10.1590/S1982-21702011000100001 (accessed 18-Feb-2013)

Sloane, N.J.A., Sequence A096427,A014549, A053004 in ‘The On-Line Encyclopedia of Integer Sequences’, http://oeis.org

Watson, G.N., 1933, ‘The Marquis and the Land-Agent: A Tale of the Eighteenth Century’, *The Mathematical Gazette,* Vol. 17, No. 222, (Feb., 1933), pp. 5-17. Also in: Berggren, L., Borwein, J. and Borwein, P., 1997, *Pi: A Source Book,* Springer, New York, pp. 258-270.